7. [12 points] For each of the following give an example, as indicated, and provide a short (one or two sentence) explanation of why your answer is correct.

a. [4 points] Give an example of an initial value problem with a linear, second-order, homogeneous differential equation for which there is no guarantee of a unique solution.

Solution: A linear, second-order, homogeneous differential equation has the form $y'' + p(t)y' + q(t)y = 0$. For there to be no guarantee of a unique solution one or both of the functions $p(t)$ and $q(t)$ must be discontinuous at the initial condition. Thus, one such example is $y'' + t^{-1}y' + y = 0$, $y(0) = y_0$, $y'(0) = v_0$.

b. [4 points] Give an example of a linear, second-order, constant-coefficient, nonhomogeneous differential for which we cannot use the method of undetermined coefficients. What form will the general solution to your equation take?

Solution: We know that undetermined coefficients fails when the forcing is not polynomial, sinusoidal, exponential or a product of these. Thus one such example is $y''+2y'+y = t^{-1}$. The solution will be of the form $y = c_1e^{-t} + c_2te^{-t} + u_1(t)e^{-t} + u_2(t)te^{-t}$, using variation of parameters.

c. [4 points] Give an example of a linear, second-order, nonhomogeneous differential equation for which the Laplace transform of the dependent variable y could be $\mathcal{L}\{y(t)\} = Y(s) = \frac{1}{s+1} + \frac{1}{s+2} + \frac{1}{s+3}$.

Solution: The equation is second-order, so the general solution will be of the form $y = c_1y_1 + c_2y_2 + y_p$. Here we see that the solution is $y = e^{-t} + e^{-2t} + e^{-3t}$. Recognizing this, there are a number of ways to proceed. The first is to identify two of the three exponentials must be part of the complementary homogeneous problem, so that the characteristic polynomial is one of $(\lambda+1)(\lambda+2) = \lambda^2+3\lambda+2$, $(\lambda+2)(\lambda+3) = \lambda^2+5\lambda+6$, or $(\lambda+1)(\lambda+3) = \lambda^2+4\lambda+3$. With these, the equation must be one of $y''+3y'+2y = 2e^{-3t}$, $y'' + 5y' + 6y = 2e^{-t}$, or $y'' + 4y' + 3y = -e^{-2t}$.

A second approach is to take this solution and create a differential equation; for example, if $y = e^{-t} + e^{-2t} + e^{-3t}$, then $y'' = e^{-t} + 4e^{-2t} + 9e^{-3t}$ is a linear, second-order nonhomogeneous differential equation for which the Laplace transform could be as indicated. With some care, it is possible to pick other solutions as well.