2. [14 points] A model for a population that is susceptible to a disease is the SI (Susceptible, Infected) model. With a few simplifying assumptions, we may model smallpox infections in a population with the SI model

$$\begin{aligned} S' &= -4SI + k(1 - S - I) \\ I' &= 4SI - I, \end{aligned}$$

where S is the fraction of the total population that is susceptible to smallpox and I is the fraction who are infected by the disease. (The remainder of the population is recovered.) We shall consider this with $k = 2$, in which case the equilibrium solutions to the system are $(S, I) = (1, 0)$ and $(S, I) = (1/4, 1/2)$.

a. [5 points] Find the linearization of this system at the critical point $(1, 0)$. Solve the linear system that you obtain.

Solution: The Jacobian for the system is $J = \begin{pmatrix} -4I - 2 & -4S - 2 \\ 4I & 4S - 1 \end{pmatrix}$, which at $(1, 0)$ is $J(1, 0) = \begin{pmatrix} -2 & -6 \\ 0 & 3 \end{pmatrix}$. Thus, if $\mathbf{u} = \begin{pmatrix} u \\ v \end{pmatrix}$ is a small displacement from $(1, 0)$, we have $\mathbf{u}' = \begin{pmatrix} -2 & -6 \\ 0 & 3 \end{pmatrix} \mathbf{u}$. Eigenvalues of $J(1, 0)$ are $\lambda = -2, 3$; for $\lambda = -2$, $\mathbf{v} = \begin{pmatrix} 1 & 0 \end{pmatrix}^T$ and for $\lambda = 3$, $\mathbf{v} = \begin{pmatrix} -6 & 5 \end{pmatrix}^T$. Thus

$$\mathbf{u} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} -6 \\ 5 \end{pmatrix} e^{3t}.$$

b. [4 points] Find the linearization of this system at the critical point $(1/4, 1/2)$. Determine the type of critical point this is (that is, whether it is a node, saddle or spiral point, and its stability).

Solution: Using the Jacobian from above and again taking $\mathbf{u}$ to be the equilibrium solution we have $\mathbf{u}' = \begin{pmatrix} -4 & -3 \\ 2 & 0 \end{pmatrix} \mathbf{u}$. Eigenvalues of the coefficient matrix are given by $(-4 - \lambda)(-\lambda) + 6 = \lambda^2 + 4\lambda + 6 = (\lambda + 2)^2 + 2 = 0$, so that $\lambda = -2 \pm i\sqrt{2}$. Thus this is an asymptotically stable spiral point.