3. [16 points] Consider a model for interacting populations x_1 and x_2 given by

$$x_1' = 2x_1 - \frac{4x_1x_2}{3+x_1}, \quad x_2' = -x_2 + \frac{2x_1x_2}{3+x_1}.$$

a. [2 points] What type of interaction do you think there is between these populations (how does the interaction affect each population)? Explain.

Solution: We expect that x_1 is a prey population and x_2 a predator. The interaction terms are $-\frac{x_1x_2}{3+x_1}$ for x_1 and $\frac{x_1x_2}{3+x_1}$ for x_2. Thus x_1 is disadvantaged by the interaction, while x_2 is advantaged. Further, in the absence of x_1, the population of x_2 will exponentially decay to zero.

b. [4 points] Find all critical points for this system.

Solution: Setting derivatives to zero, the second equation gives $x_2(-1 + \frac{2x_1}{3+x_1}) = 0$, so $x_2 = 0$ or $2x_1 = 3 + x_1$, so $x_1 = 3$. If $x_2 = 0$, the first equation requires $x_1 = 0$. If $x_1 = 3$, the first equation becomes $0 = 6 - 2x_2$, so that $x_2 = 3$. Thus there are two critical points, $(0, 0)$ and $(3, 3)$.

c. [6 points] The Jacobian for this system is $J(x_1, x_2) = \begin{pmatrix} 2 - \frac{12x_2}{(x_1+3)^2} & -\frac{4x_1}{x_1+3} \\ \frac{6x_2}{(x_1+3)^2} & -1 + \frac{2x_1}{x_1+3} \end{pmatrix}$. Classify each of your critical points from (b) by stability and type, and sketch a phase portrait for each. *(This problem part continues on the next page.)*

Problem 3, continued. We are considering the system

$$x_1' = 2x_1 - \frac{4x_1x_2}{3+x_1}, \quad x_2' = -x_2 + \frac{2x_1x_2}{3+x_1}.$$

c. *Continued: we are solving the problem*

The Jacobian for this system is $J(x_1, x_2) = \begin{pmatrix} 2 - \frac{12x_2}{(x_1+3)^2} & -\frac{4x_1}{x_1+3} \\ \frac{6x_2}{(x_1+3)^2} & -1 + \frac{2x_1}{x_1+3} \end{pmatrix}$. Classify each of your critical points from (b) by stability and type, and sketch a phase portrait for each.

Solution: From the given Jacobian, we have at at the critical point $(0, 0)$, $J(0, 0) = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}$, so that $\lambda = 2, -1$, and we see that the origin is an unstable saddle point.

At $(3, 3)$, we have $J(3, 3) = \begin{pmatrix} 1 & -2 \\ \frac{1}{2} & 0 \end{pmatrix}$, so that eigenvalues satisfy $\lambda^2 - \lambda + 1 = (\lambda - \frac{1}{2})^2 + \frac{3}{4} = 0$, and $\lambda = \frac{1}{2} \pm \frac{\sqrt{3}}{2}\, i$. Thus this is unstable spiral source.

(Phase portraits are omitted here, but are as expected.)

d. [4 points] If the population of x_1 was initially large and that of x_2 small, sketch a qualitatively accurate graph of x_1 and x_2 as functions of time. What happens to the populations for large times?

Solution: Because of the context, we would initially guess that the population of x_1 will decrease and that of x_2 will increase, and that the populations will oscillate with decreasing magnitude as they converge to $x_1 = x_2 = 3$. However, this isn't what our analysis above suggests: because the critical point $(3, 3)$ is an unstable spiral, we expect that the trajectory will either move around $(3, 3)$ (possibly more than once) before heading out along the x_1 axis and increasing to (∞, ∞), or will fail to oscillate at all and move to the same limiting behavior.

(The solution curves are graphs of x_1 and x_2 against time.)