5. [12 points] Consider the nonlinear system

\[ x' = y, \quad y' = -3x - 2y + rx^2. \]

Four possible phase portraits for this system are shown along the right side of the page.

a. [4 points] If one of the graphs is to match this system, what is the value of the parameter \( r \)? Why?

**Solution:** We note that critical points for the system are given by \( y = 0, -3x - 2y + rx^2 = 0 \), or \( x(-3 + rx) = 0 \). Thus the critical points are \((0,0)\) and \((3/r,0)\). From the phase portraits we know that the critical points are \((0,0)\) and \((1,0)\), so \( r = 3 \).

b. [8 points] Given the value of \( r \) you found in (a), which, if any, of the phase portraits could be that for this system? Why?

**Solution:** This is most easily determined by finding the linear behavior near the critical points. The Jacobian for the system is \( J = \begin{pmatrix} 0 & 1 \\ -3 + 6x & -2 \end{pmatrix} \), so at \((0,0)\) we have \( J(0,0) = \begin{pmatrix} 0 & 1 \\ -3 & -2 \end{pmatrix} \), and at \((1,0)\), \( J(1,0) = \begin{pmatrix} 0 & 1 \\ 3 & -2 \end{pmatrix} \). The eigenvalues of the corresponding systems are given by \( \lambda^2 + 2\lambda + 3 = 0 \). For \((0,0)\), \( \lambda^2 + 2\lambda + 3 = (\lambda + 1)^2 + 2 = 0 \), so \( \lambda = -1 \pm i\sqrt{2} \), and the critical point is a spiral sink. We note also that to the right of \((0,0)\), \( u' = 0 \) and \( v' < 0 \), so rotation is clockwise.

For \((1,0)\), \( \lambda^2 + 2\lambda - 3 = (\lambda + 3)(\lambda - 1) = 0 \), so that \( \lambda = -3 \) and \( \lambda = 1 \), and the point is a saddle point. We note also that when \( \lambda = -3 \), \( \mathbf{v} = \begin{pmatrix} 1 \\ -3 \end{pmatrix} \) and when \( \lambda = 1 \), \( \mathbf{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \).

Thus phase portrait 3 could be correct.