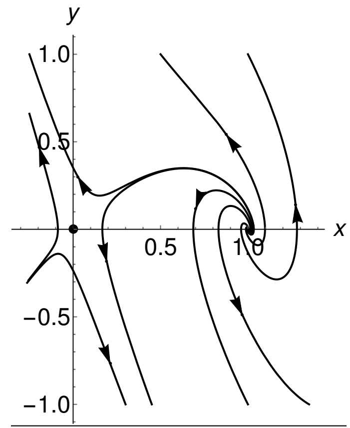
5. [12 points] Consider the nonlinear system

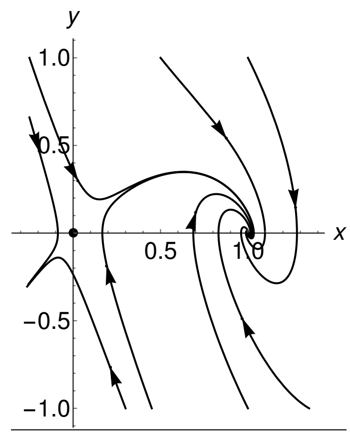
$$x' = y, \quad y' = -3x - 2y + r\,x^2.$$

Four possible phase portraits for this system are shown along the right side of the page.

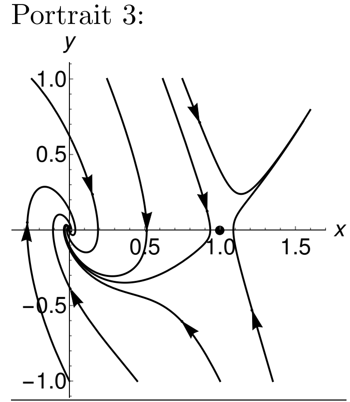
Portrait 1:



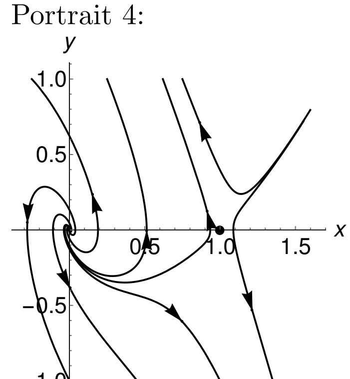
Portrait 2:



Portrait 3:



Portrait 4:



a. [4 points] If one of the graphs is to match this system, what is the value of the parameter r? Why?

Solution: We note that critical points for the system are given by $y = 0$, $-3x - 2y + r\,x^2 = 0$, or $x(-3 + r\,x) = 0$. Thus the critical points are $(0, 0)$ and $(3/r, 0)$. From the phase portraits we know that the critical points are $(0, 0)$ and $(1, 0)$, so $r = 3$.

b. [8 points] Given the value of r you found in **(a)**, which, if any, of the phase portraits could be that for this system? Why?

Solution: This is most easily determined by finding the linear behavior near the critical points. The Jacobian for the system is $J = \begin{pmatrix} 0 & 1 \\ -3+6x & -2 \end{pmatrix}$, so at $(0, 0)$ we have $J(0,0) = \begin{pmatrix} 0 & 1 \\ -3 & -2 \end{pmatrix}$, and at $(1, 0)$, $J(1,0) = \begin{pmatrix} 0 & 1 \\ 3 & -2 \end{pmatrix}$. The eigenvalues of the corresponding systems are given by $\lambda^2 + 2\lambda \pm 3 = 0$. For $(0, 0)$, $\lambda^2 + 2\lambda + 3 = (\lambda+1)^2 + 2 = 0$, so $\lambda = -1 \pm i\sqrt{2}$, and the critical point is a spiral sink. We note also that to the right of $(0, 0)$, $u' = 0$ and $v' < 0$, so rotation is clockwise.
For $(1, 0)$, $\lambda^2 + 2\lambda - 3 = (\lambda+3)(\lambda-1) = 0$, so that $\lambda = -3$ and $\lambda = 1$, and the point is a saddle point. We note also that when $\lambda = -3$, $\mathbf{v} = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$ and when $\lambda = 1$, $\mathbf{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Thus phase portrait 3 could be correct.