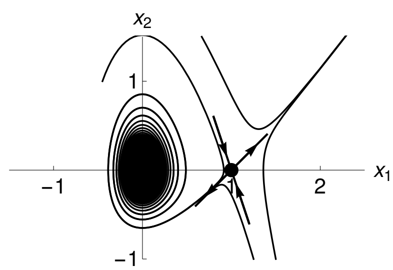
8. [12 points] Suppose that for some nonlinear second-order differential equation $y'' = f(y)$ we can write an equivalent system of two first-order differential equations $x_1' = F(x_1, x_2)$, $x_2' = G(x_1, x_2)$. Critical points of the latter are $\mathbf{x}_0 = (0,0)$ and $\mathbf{x}_1 = (1,0)$. The Jacobian at these points is $\mathbf{J}(\mathbf{x}_0) = \begin{pmatrix} 0 & 1 \\ -3 & -2 \end{pmatrix}$ and $\mathbf{J}(\mathbf{x}_1) = \begin{pmatrix} 0 & 1 \\ 3 & -2 \end{pmatrix}$.

a. [8 points] Sketch a phase portrait for the nonlinear system.

Solution: At $\mathbf{x}_0$ the Jacobian has eigenvalues given by $\lambda^2 + 2\lambda + 3 = (\lambda+1)^2 + 2 = 0$, so that $\lambda = -1 \pm i\sqrt{2}$, and $\mathbf{x}_0$ is a stable spiral point. Note that (considering the point $(1,0)$, where the derivatives on a trajectory are $(0,-3)$) the spiral moves in a clockwise direction.
At $\mathbf{x}_1$, the Jacobian has eigenvalues given by $\lambda^2 + 2\lambda - 3 = (\lambda+1)^2 - 4 = 0$, so $\lambda = -3$ or $\lambda = 1$. If $\lambda = -3$, $\mathbf{v} = \begin{pmatrix} 1 & -3 \end{pmatrix}^T$, and if $\lambda = 1$, $\mathbf{v} = \begin{pmatrix} 1 & 1 \end{pmatrix}^T$.
Plotting these and adding the expected trajectories gives the phase portrait below.



b. [4 points] Based on your phase portrait, sketch a qualitatively accurate graph of y as a function of t if we start with the initial condition $y(0) = 0$, $y'(0) = 1$.

Solution: From the phase portrait, we see that a trajectory starting at $(0,1)$ should spiral in to the origin. Thus we know that $y(0) = 0$ and $y'(0) = 1$, and that y must be a decaying sinusoidal function. Thus we have a graph like that shown below.