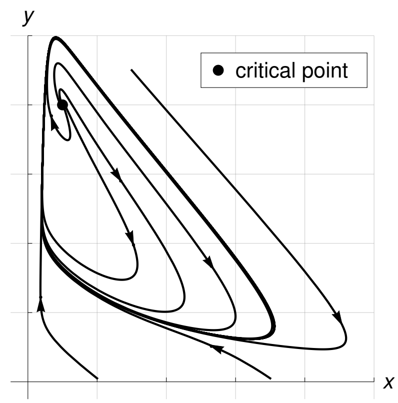
9. [14 points] The Brusselator is a nonlinear model of a chemical reaction which can have oscillatory concentrations x and y of the chemicals in the reaction. A model for this is

$$x' = 1 - (b+1)x + \frac{1}{4}x^2y, \quad y' = bx - \frac{1}{4}x^2y.$$



The figure to the right gives the phase portrait for this system for some value of b.

a. [4 points] What are the coordinates of the critical point shown? *(Note that your answer may involve the parameter b.)*

Solution: Critical points are where $x' = 1 - x((b+1) - \frac{1}{4}xy) = 0$ and $y' = x(b - \frac{1}{4}xy) = 0$. The latter says that $x = 0$ or $xy = 4b$. If $x = 0$, the first is unsolvable. If $xy = 4b$, the first becomes $1 - x(b+1-b) = 0$, so $x = 1$ and $y = 4b$. The critical point is $(x, y) = (1, 4b)$.

b. [7 points] Given the behavior shown in the phase portrait, what can you say about the parameter b?

Solution: The phase portrait shows that the linear behavior of the critical point is that of an unstable spiral point. To see how b may determine this, we find the Jacobian and investigate the behavior at the critical point. The Jacobian is

$$J = \begin{pmatrix} -(b+1) + \frac{1}{2}xy & \frac{1}{4}x^2 \\ b - \frac{1}{2}xy & -\frac{1}{4}x^2 \end{pmatrix}\Bigg|_{x=1, y=4b} = \begin{pmatrix} b-1 & \frac{1}{4} \\ -b & -\frac{1}{4} \end{pmatrix}.$$

The behavior of the critical point will be determined by the eigenvalues of this Jacobian, which are given by $(b - 1 - \lambda)(-\frac{1}{4} - \lambda) + \frac{1}{4}b = \lambda^2 + (\frac{5}{4} - b)\lambda + \frac{1}{4} = 0$, or

$$\lambda = \frac{1}{2}(b - \frac{5}{4}) \pm \frac{1}{2}\sqrt{(b - \frac{5}{4})^2 - 1}.$$

For this to give an unstable spiral, we must have $b > \frac{5}{4}$, and $(b - \frac{5}{4})^2 - 1 < 0$. The latter says that $-1 < (b - \frac{5}{4}) < 1$, so we know that $\frac{5}{4} < b < \frac{9}{4}$.
It is possible from the picture that we have real eigenvalues, but we still know that the point must be unstable. If that were the case we would require $b > \frac{5}{4}$ and $(b - \frac{5}{4})^2 \geq 1$, so that $b \geq \frac{9}{4}$.

c. [3 points] We said that the Brusselator can have oscillatory concentrations of x and y. Explain how the result here does (or does not) demonstrate this behavior.

Solution: We see that the critical point is unstable, and trajectories converge to something that looks like the limit cycle that we explored in the van der Pol equation in lab 2. On that limit cycle, the values of x and y will oscillate as we move around the closed trajectory.