1. [12 points] Suppose we are solving the linear system $\mathbf{x}' = \mathbf{A}\mathbf{x} + \begin{pmatrix} 0 \\ 8 \end{pmatrix}$.

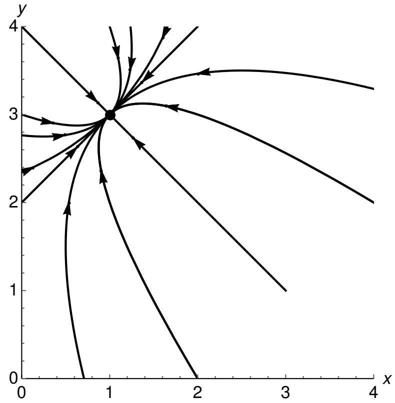
a. [4 points] If $\mathbf{A} = \begin{pmatrix} -3 & 1 \\ 1 & -3 \end{pmatrix}$, find all critical points for the system.

Solution: These are where $\mathbf{x}$ is a constant, so that, with $\mathbf{x} = \begin{pmatrix} x & y \end{pmatrix}^T$, $-3x + y = 0$ and $x - 3y = -8$. Thus $-8x = -8$, and $x = 1$, so that $y = 3$. The only critical point is $(1, 3)$.

b. [5 points] If the eigenvalues and eigenvectors of $\mathbf{A}$ are $\lambda_{1,2} = -4, -2$ with $\mathbf{v}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, sketch a phase portrait for the system.

Solution: From our work in (a) we know that the critical point for this system is $\mathbf{x}_c = (1, 3)$. Then, either by letting $\mathbf{x} = \mathbf{x}_c + \mathbf{u}$ (so that $\mathbf{u}' = \mathbf{A}\mathbf{u}$) or by noting that the general solution is $\mathbf{x} = \mathbf{x}_c + \mathbf{x}_p = \mathbf{u} + \mathbf{x}_c$, we know that the phase portrait for the homogeneous system $\mathbf{u}' = \mathbf{A}\mathbf{u}$ will be centered on $\mathbf{x}_c$.

The phase portrait there will have two straight line solutions that correspond to the eigenvectors, with solutions converging to $\mathbf{x}_c$. The eigenvector associated with $\mathbf{v}_1^T = \begin{pmatrix} -1 & 1 \end{pmatrix}$ is much smaller (more negative) than the other, so trajectories will collapse in this direction first, and then converge to the critical point along $\mathbf{v}_2^T = \begin{pmatrix} 1 & 1 \end{pmatrix}$, giving the phase portrait below.



c. [3 points] For a different $\mathbf{A}$, could a solution to the system be $x = e^{-3t}\sin(t)$, $y = e^{-3t}\cos(t)$? Explain.

Solution: This is not possible, because of the shift in the critical point induced by the inhomogeneity. If we had $x = e^{-3t}\sin(t) + 1$, $y = e^{-3t}\cos(t) + 3$, following the logic given in the first paragraph in the solution for part (b), it could be the solution for a different matrix $\mathbf{A}$.