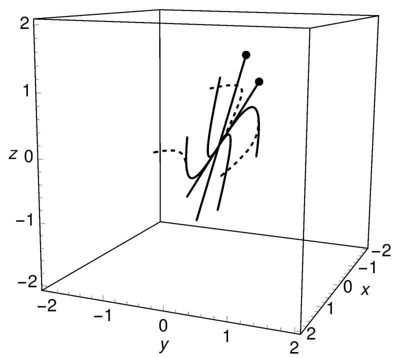
2. [10 points] In this problem we consider a linearization of the Lorenz system that we considered in lab 5, with $\eta = \sqrt{8(r-1)/3}$,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}' = \begin{pmatrix} -10 & 10 & 0 \\ 1 & -1 & -\eta \\ \eta & \eta & -2.67 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

a. [5 points] For some value of r, if phase portrait is shown in the figure to the right, below, what can you say about the eigenvalues and eigenvectors of this system (please, whatever you do, do not try to calculate exact values from the system)? The solid black trajectories lie in a plane. The dashed trajectories start to the left the plane as you look at it. The two black points are, approximately, $(1, 0.75, 1.5)$ and $(1, 1, 1.25)$. You should be able to specify at least two eigenvectors and the relative values of the eigenvalues.



Solution: Note that all eigenvalues must be real, because there are two straight line trajectories (and thus there must be three). We assume that trajectories decay to the origin. Then, because the dashed trajectories decay rapidly into the plane determined by the solid trajectories, we know that one eigenvalue is much more negative than the other two. Given the two points, which appear to lie on straight-line solutions in the plane, we know that two eigenvectors are $\mathbf{v}_1 = \begin{pmatrix} 1 & 0.75 & 1.5 \end{pmatrix}^T$ and $\mathbf{v}_2 = \begin{pmatrix} 1 & 1 & 1.25 \end{pmatrix}^T$ (note that $\mathbf{v}_1$ has the larger z-coordinate), with eigenvalues $\lambda_1 < \lambda_2 < 0$. The third eigenvalue, λ_3, is significantly more negative than λ_1.

If we assume solutions grow instead of decaying we have the same analysis, but with $0 < \lambda_2 < \lambda_1 < \lambda_3$.

b. [5 points] For a different value of r, the general solution to the system is, approximately,

$$\begin{aligned} x &= c_1 e^{-12t} + c_2\,(0.5\,\cos(0.4\,t) - 0.05\,\sin(0.4\,t))\,e^{-t} + \\ &\qquad c_3\,(0.05\,\cos(0.4\,t) + 0.5\,\sin(0.4\,t))\,e^{-t} \\ y &= -0.4\,c_1\,e^{-12t} + c_2\,(0.5\,\cos(0.4\,t) - 0.05\,\sin(0.4\,t))\,e^{-t} + \\ &\qquad c_3\,(0.05\,\cos(0.4\,t) + 0.5\,\sin(0.4\,t))\,e^{-t} \\ z &= -0.1\,c_1\,e^{-12t} + 0.75\,c_2\,e^{-t}\cos(0.4\,t) + 0.75\,c_3\,e^{-t}\sin(0.4\,t). \end{aligned}$$

What are the eigenvalues and eigenvectors of the coefficient matrix for the linear system in this case?

Solution: We see that $\lambda_1 = -12$ and $\lambda_{2,3} = -1 \pm 0.4i$. The eigenvector $\mathbf{v}_1$ we can read from the first component of the solutions: $\mathbf{v}_1 = \begin{pmatrix} 1 & -0.4 & -0.1 \end{pmatrix}^T$. For the complex λ, we can read the real and imaginary parts of each component of the vector from the coefficient of the cosine terms in the second and third solutions, getting $\mathbf{v}_{2,3} = \begin{pmatrix} 0.5 \pm 0.05i & 0.5 \pm 0.05i & 0.75 \end{pmatrix}^T$.