3. [12 points] In each of the following we consider a first order differential equation \( y' = f(t, y) \).
In these, the functions \( f(t, y) \) and \( g(t, y) \) are different functions.

a. [6 points] The direction field for the equation \( y' = f(t, y) \) is shown to the right. For each of the follow-
ing, explain if the statement is true, false, or if you
cannot tell.

(1) The equation is autonomous, that is, \( f(t, y) \) is
actually only a function of \( y \).
(2) The equation is linear.
(3) The initial value problem \( y' = f(t, y) \), \( y(0) = y_0 \)
has a unique solution for all \( y_0 \) between \(-2\) and 2.

**Solution:**
(1) The equation is not autonomous,
because the derivative \( y' \) depends on \( t \)
(2) The equation is not linear: if it where, we would have \( y' = -p(t)y + q(t) \), and
for fixed \( t \) the could be only one critical point (which is not true; for example, along
\( t = 2 \) we have critical points at \( y = 0 \) and \( y = 2 \)).
(3) There are no points \((0, y_0)\) at which the function value (slope shown in the direction
field) is not continuously changing from the previous value, and the change (derivative
\( f_y \)) is similarly continuous, so there should be a unique solution for all of these points.

b. [6 points] Let \( y' = g(t, y) = y(y^3 - a^3) \), where \( a \) is a real number. Identify all \( a \) for which
it is true both there is a critical point other than \( y = 0 \), and that \( y = 0 \) is stable. Be
sure it is clear how you arrive at your conclusion. Draw a phase line for this situation, or
explain why it is impossible.

**Solution:**
If \( a = 0 \), we have \( y' = y^4 \), and the only critical point is \( y = 0 \). Note that
in this case \( y' > 0 \) for all \( y \neq 0 \), so the critical point is unstable (or, if one likes the
term, semi-stable). If \( a \neq 0 \), there are two critical points, \( y = 0 \) and \( y = a \). If \( a < 0 \),
the equation is \( y' = y(y^3 + |a|^3) \). For \( y > 0 \), \( y' > 0 \); for \(-|a| < y < 0 \), \( y' < 0 \), and for
\( y < -|a| \) \( y' > 0 \). Thus when \( a < 0 \), the critical point \( y = 0 \) is unstable and \( y = -|a| \) is
stable. Similarly, if \( a > 0 \), \( y' > 0 \) when \( y < 0 \); \( y' < 0 \) when \( 0 < y < a \); and \( y' > 0 \) when
\( y > a \). Thus \( y = 0 \) is stable and \( y = a \) is unstable. The values of \( a \) we want are \( a > 0 \).
The phase line is shown below.