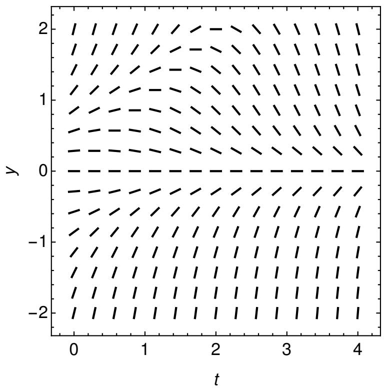
3. [12 points] In each of the following we consider a first order differential equation $y' = f(t, y)$. In these, the functions $f(t, y)$ and $g(t, y)$ are different functions.

a. [6 points] The direction field for the equation $y' = f(t, y)$ is shown to the right. For each of the following, explain if the statement is true, false, or if you cannot tell.

(1) The equation is autonomous, that is, $f(t, y)$ is actually only a function of y.

(2) The equation is linear.

(3) The initial value problem $y' = f(t, y)$, $y(0) = y_0$ has a unique solution for all y_0 between -2 and 2.



Solution: (1) The equation is not autonomous, because the derivative y' depends on t

(2) The equation is not linear: if it where, we would have $y' = -p(t)y + q(t)$, and for fixed t the could be only one critical point (which is not true; for example, along $t = 2$ we have critical points at $y = 0$ and $y = 2$).

(3) There are no points $(0, y_0)$ at which the function value (slope shown in the direction field) is not continuously changing from the previous value, and the change (derivative f_y) is similarly continuous, so there should be a unique solution for all of these points.

b. [6 points] Let $y' = g(t, y) = y(y^3 - a^3)$, where a is a real number. Identify all a for which it is true both there is a critical point other than $y = 0$, and that $y = 0$ is stable. Be sure it is clear how you arrive at your conclusion. Draw a phase line for this situation, or explain why it is impossible.

Solution: If $a = 0$, we have $y' = y^4$, and the only critical point is $y = 0$. Note that in this case $y' > 0$ for all $y \neq 0$, so the critical point is unstable (or, if one likes the term, semi-stable). If $a \neq 0$, there are two critical points, $y = 0$ and $y = a$. If $a < 0$, the equation is $y' = y(y^3 + |a|^3)$. For $y > 0$, $y' > 0$; for $-|a| < y < 0$, $y' < 0$, and for $y < -|a|$ $y' > 0$. Thus when $a < 0$, the critical point $y = 0$ is unstable and $y = -|a|$ is stable. Similarly, if $a > 0$, $y' > 0$ when $y < 0$; $y' < 0$ when $0 < y < a$; and $y' > 0$ when $y > a$. Thus $y = 0$ is stable and $y = a$ is unstable. The values of a we want are $a > 0$. The phase line is shown below.