6. [12 points] Consider the nonlinear system

$$x' = 1 - y, \qquad y' = 2 - 2y + 3\sin(x).$$

Sketch a qualitatively accurate phase portrait showing representative trajectories, by doing appropriate linearization and local analysis. Use your phase portrait to predict the behavior of a trajectory starting at $x(0) = \pi$, $y(0) = 0$.

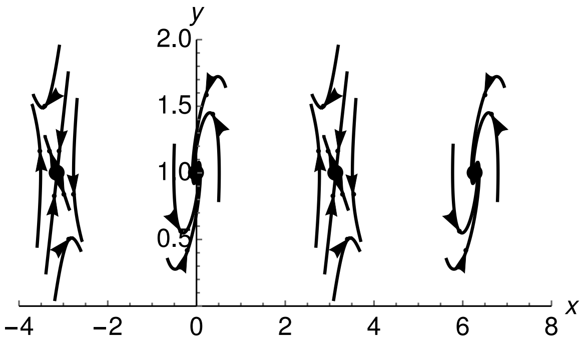
Solution: To sketch the phase portrait, we first find critical points. Requiring $x' = 0 = 1 - y$, we have $y = 1$. Then $y' = 0 = 2 - 2(1) + 3\sin(x)$, so $x = n\pi$, for any integer n. To determine the linear behavior at each of the critical points $(n\pi, 1)$, we find the Jacobian to tell us the coefficient matrices for the linear systems. We have $J(x, y) = \begin{pmatrix} 0 & -1 \\ 3\cos(x) & -2 \end{pmatrix}$. Thus, at $(n\pi, 1)$ with n even and odd, the coefficient matrices are, respectively,

$$J_{2n} = \begin{pmatrix} 0 & -1 \\ 3 & -2 \end{pmatrix} \quad \text{and} \quad J_{2n+1} = \begin{pmatrix} 0 & -1 \\ -3 & -2 \end{pmatrix}.$$

At $x = n\pi$, n even, eigenvalues of the coefficient matrix satisfy $\lambda^2 + 2\lambda + 3 = (\lambda + 1)^2 + 2 = 0$, so $\lambda = -1 \pm i\sqrt{2}$. Thus at these critical points we have a spiral sink. Further, starting from $(1, 0)$ then gives $(x', y') = (0, 3)$, so motion is counter-clockwise around the critical points.

At $x = n\pi$, n odd, eigenvalues of the coefficient matrix satisfy $\lambda^2 + 2\lambda - 3 = (\lambda + 3)(\lambda - 1) = 0$, so $\lambda = -3$ or $\lambda = 1$, and these are unstable saddle points. The eigenvectors associated with the eigenvalues are, respectively, $\mathbf{v}_{-3} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ and $\mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

Putting these together, we have the phase portrait shown below.



From this, we see that an initial condition $(\pi, 0)$ will follow the saddle trajectory below the critical point $(\pi, 1)$ up and to the right, then down, and will likely spiral in to either the critical point $(2\pi, 1)$. This gives the dashed trajectory shown below.