6. [12 points] Consider the nonlinear system
\[ x' = 1 - y, \quad y' = 2 - 2y + 3\sin(x). \]
Sketch a qualitatively accurate phase portrait showing representative trajectories, by doing appropriate linearization and local analysis. Use your phase portrait to predict the behavior of a trajectory starting at \( x(0) = \pi, \ y(0) = 0 \).

Solution: To sketch the phase portrait, we first find critical points. Requiring \( x' = 0 = 1 - y \), we have \( y = 1 \). Then \( y' = 0 = 2 - 2(1) + 3\sin(x) \), so \( x = n\pi \), for any integer \( n \). To determine the linear behavior at each of the critical points \((n\pi, 1)\), we find the Jacobian to tell us the coefficient matrices for the linear systems. We have \( J(x, y) = \begin{pmatrix} 0 & -1 \\ 3\cos(x) & -2 \end{pmatrix} \). Thus, at \((n\pi, 1)\) with \( n \) even and odd, the coefficient matrices are, respectively,
\[
J_{2n} = \begin{pmatrix} 0 & -1 \\ 3 & -2 \end{pmatrix} \quad \text{and} \quad J_{2n+1} = \begin{pmatrix} 0 & -1 \\ -3 & -2 \end{pmatrix}.
\]
At \( x = n\pi, n \) even, eigenvalues of the coefficient matrix satisfy \( \lambda^2 + 2\lambda + 3 = (\lambda + 1)^2 + 2 = 0 \), so \( \lambda = -1 \pm i\sqrt{2} \). Thus at these critical points we have a spiral sink. Further, starting from \((1, 0)\) then gives \((x', y') = (0, 3)\), so motion is counter-clockwise around the critical points.
At \( x = n\pi, n \) odd, eigenvalues of the coefficient matrix satisfy \( \lambda^2 + 2\lambda - 3 = (\lambda + 3)(\lambda - 1) = 0 \), so \( \lambda = -3 \) or \( \lambda = 1 \), and these are unstable saddle points. The eigenvectors associated with the eigenvalues are, respectively, \( v_{-3} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \) and \( v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \).
Putting these together, we have the phase portrait shown below.

![Phase portrait](image)

From this, we see that an initial condition \((\pi, 0)\) will follow the saddle trajectory below the critical point \((\pi, 1)\) up and to the right, then down, and will likely spiral in to either the critical point \((2\pi, 1)\). This gives the dashed trajectory shown below.