1. [12 points] Consider the system of differential equations $\mathbf{x}' = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 3 & -4 \end{pmatrix} \mathbf{x}$.

a. [6 points] Find the general solution to this system.[1]

Solution: Finding the eigenvalues of the coefficient matrix, we have $\det(\mathbf{A} - \lambda\mathbf{I}) = (-2-\lambda)((1-\lambda)(-4-\lambda)+6) = (-2-\lambda)(\lambda^2+3\lambda+2) = (-2-\lambda)(\lambda+2)(\lambda+1) = 0$, so that $\lambda = -1$ or $\lambda = -2$ (repeated). Then, if $\lambda = -1$, we have for the eigenvector $\begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & -2 \\ 0 & 3 & -3 \end{pmatrix} \mathbf{v} = \mathbf{0}$, so that $\mathbf{v} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$. If $\lambda = -2$, $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & -2 \\ 0 & 3 & -2 \end{pmatrix} \mathbf{v} = \mathbf{0}$, so that we may take $\mathbf{v} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ or $\mathbf{v} = \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix}$. Thus the general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^{-2t} + c_3 \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix} e^{-2t}.$$

b. [6 points] Now suppose that we consider only initial conditions in the yz-plane (that is, we take $\mathbf{x}(0) = \begin{pmatrix} 0 \\ y_0 \\ z_0 \end{pmatrix}$). Sketch the phase portrait for these initial conditions, in the yz-plane.

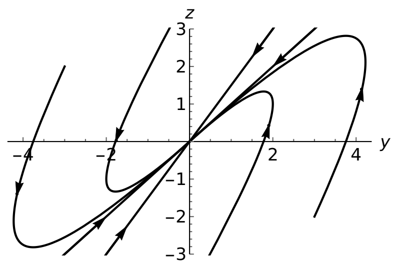
Solution: If we are restricted to the yz-plane, we have only the first and last terms in the general solution,

$$\mathbf{x} = c_1 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} e^{-t} + c_3 \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix} e^{-2t},$$

so that in the yz-plane, we have

$$\begin{pmatrix} y \\ z \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} + c_3 \begin{pmatrix} 2 \\ 3 \end{pmatrix} e^{-2t}.$$

We can sketch the phase portrait in this plane by drawing in the eigenvectors $z = y$ and $z = \frac{3}{2}y$ and the corresponding trajectories, which collaps to the first eigenvector and then to the origin. This is shown below.



[1]Possibly useful: $\det\begin{pmatrix} a & 0 & 0 \\ 0 & b & c \\ 0 & d & e \end{pmatrix} = a(be - cd)$.