4. [12 points] Consider the system of differential equations given by $x' = Ax$, where $A$ is a real-valued $2 \times 2$ matrix and $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.

a. [6 points] Suppose that the eigenvalues and eigenvectors of $A$ are $\lambda = -1 \pm i$, with $v = \begin{pmatrix} 2 \pm i \\ 1 \end{pmatrix}$. If $x$ solves $x(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, sketch the trajectory for $x$ in the phase plane.

Solution: The general solution is $x = c_1 \begin{pmatrix} 2 \cos(t) - \sin(t) \\ \cos(t) \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} \cos(t) + 2 \sin(t) \\ \sin(t) \end{pmatrix} e^{-t}$, so to satisfy the initial condition we may take $c_1 = 0$ and $c_2 = 1$. Then, as $t$ increases we see that $y$ component of the trajectory increases, and it will be an inward spiral, giving the trajectory shown below.

(If we want to outdo ourselves, we could also note that $x'(0) = \begin{pmatrix} 1 & 1 \end{pmatrix}^T$ to give the initial direction of the trajectory, thereby getting the orientation of the spiral shown above.)

b. [6 points] Suppose that eigenvalues and eigenvectors of $A$ are $\lambda_1 = 1$ and $\lambda_2 = 2$, with $v_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $v_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$. If $x(0) = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$, as $t \to \infty$, which of the following is most correct, and why? (i) $x_2 \approx 2x_1$; (ii) $x_2 \approx -\frac{1}{2}x_1$; (iii) $x_2 \approx -\frac{1}{2} x_1 - 1$; (iv) $x_2 \approx -\frac{1}{2} x_1 - k$, with $k > 1$.

Solution: The phase portrait for the system will look something like the following.

All trajectories end up parallel to the second eigenvector, which has slope $m = -\frac{1}{2}$. Looking at the dashed trajectory, we see that a trajectory through $(0, -1)$, will be shifted significantly below either of $x_2 = -\frac{1}{2} x_1$ or $x_2 = -\frac{1}{2} x_1 - 1$, so (iv) must be correct.