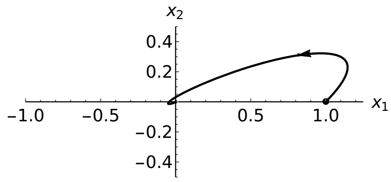
4. [12 points] Consider the system of differential equations given by $\mathbf{x}' = \mathbf{A}\mathbf{x}$, where $\mathbf{A}$ is a real-valued 2×2 matrix and $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.

a. [6 points] Suppose that the eigenvalues and eigenvectors of $\mathbf{A}$ are $\lambda = -1 \pm i$, with $\mathbf{v} = \begin{pmatrix} 2 \pm i \\ 1 \end{pmatrix}$. If $\mathbf{x}$ solves $\mathbf{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, sketch the trajectory for $\mathbf{x}$ in the phase plane.

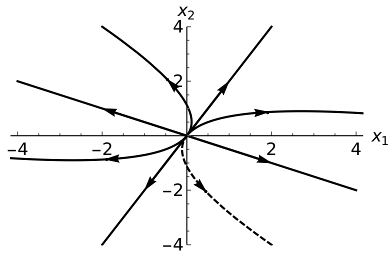
Solution: The general solution is $\mathbf{x} = c_1 \begin{pmatrix} 2\cos(t) - \sin(t) \\ \cos(t) \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} \cos(t) + 2\sin(t) \\ \sin(t) \end{pmatrix} e^{-t}$, so to satisfy the initial condition we may take $c_1 = 0$ and $c_2 = 1$. Then, as t increases we see that y component of the trajectory increases, and it will be an inward spiral, giving the trajectory shown below.



(If we want to outdo ourselves, we could also note that $x'(0) = \begin{pmatrix} 1 & 1 \end{pmatrix}^T$ to give the initial direction of the trajectory, thereby getting the orientation of the spiral shown above.)

b. [6 points] Suppose that eigenvalues and eigenvectors of $\mathbf{A}$ are $\lambda_1 = 1$ and $\lambda_2 = 2$, with $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$. If $\mathbf{x}(0) = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$, as $t \to \infty$, which of the following is most correct, and why? (i) $x_2 \approx 2x_1$; (ii) $x_2 \approx -\frac{1}{2}x_1$; (iii) $x_2 \approx -\frac{1}{2}x_1 - 1$; (iv) $x_2 \approx -\frac{1}{2}x_1 - k$, with $k > 1$.

Solution: The phase portrait for the system will look something like the following.



All trajectories end up parallel to the second eigenvector, which has slope $m = -\frac{1}{2}$. Looking at the dashed trajectory, we see that a trajectory through $(0, -1)$, will be shifted significantly below either of $x_2 = -\frac{1}{2}x_1$ or $x_2 = -\frac{1}{2}x_1 - 1$, so (iv) must be correct.