5. [12 points] In lab 6 we considered the Fitzhugh-Nagumo model for the behavior of a neuron,

$$v' = v - \frac{1}{3}v^3 - w + I_{ext}, \qquad \tau w' = v + a - bw.$$

In this problem we analyze this with the parameters $\tau = 1$, $a = \frac{1}{3}$, and $b = 1$.

a. [3 points] Find the v- and w-nullclines, and show that there is a single critical point (v_c, w_c) in this case. Find the critical point in terms of the externally applied voltage I_{ext}.

Solution: The w-nullcline occurs when $w' = 0$, and thus is $w = v + \frac{1}{3}$, The v-nullcline is when $v' = 0$, so when $0 = v - \frac{1}{3}v^3 - w + I_{ext}$. Plugging in for w, we have $0 = -\frac{1}{3}v^3 - \frac{1}{3} + I_{ext}$, so that the critical point is when $v = v_c = \sqrt[3]{3I_{ext} - 1}$ and $w = w_c = \frac{1}{3} + v_c$.

b. [3 points] Linearize the system at the critical point (write your linearization in terms of v_c and w_c—do not plug in the values you found for v_c and w_c). How is the solution to your linearized system related to the solution of the original nonlinear system?

Solution: The Jacobian of the system is $\mathbf{J} = \begin{pmatrix} 1 - v^2 & -1 \\ 1 & -1 \end{pmatrix}$, so $\mathbf{J}(v_c, w_c) = \begin{pmatrix} 1 - v_c^2 & -1 \\ 1 & -1 \end{pmatrix}$ and with $(v, w) = (v_c, w_c) + (x, y)$ the linearization is

$$\mathbf{x}' = \begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 1 - v_c^2 & -1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

The solution (x, y) to the linear system tells us the behavior of the nonlinear system when trajectories are sufficiently near the critical point (v_c, w_c). (Note that because the nonlinearity is strictly polynomial, we know that the system is almost linear and the linearization therefore makes sense.)

c. [6 points] Show that the critical point is in this case is always stable. Determine any values of v_c or w_c at which the behavior at the critical point changes. Explain how this result is different from that which you saw in lab.

Solution: The eigenvalues of $\mathbf{J}(v_c, w_c)$ are given by $\det(\mathbf{J}(v_c, w_c) - \lambda\mathbf{I}) = 0$, or $(1 - v_c^2 - \lambda)(-1 - \lambda) + 1 = \lambda^2 + v_c^2\lambda + v_c^2 = 0$. Thus, using the quadratic formula, $\lambda = -\frac{1}{2}v_c^2 \pm \frac{1}{2}\sqrt{v_c^4 - 4v_c^2} = -\frac{1}{2}v_c^2 \pm \frac{1}{2}v_c^2\sqrt{1 - 4v_c^{-2}}$. Because of the square, the leading term is always negative. Then note that $4v_c^{-2} > 0$, so the square root is either real and less than one or complex. Thus the eigenvalues are either both real and negative (if $|v_c| > 2$) or complex with negative real part (if $|v_c| < 2$); at $|v_c| = 2$ the behavior changes from a stable node to a stable spiral. This is different from what we saw in lab, because in that case there was a value of v_c (and hence I_{ext}) at which the critical point went from stable to unstable as well.

Note that the above is predicated on the assumption that $v_c \neq 0$. If $v_c = 0$, we have the degenerate case $\lambda = 0$, twice. This indicates that the linear system is stable (but not asymptotically stable), and in this case we aren't able to speak to the stability of the nonlinear system.

