6. [12 points] Consider the nonlinear system

$$x' = 3x - y - x^2, \qquad y' = -\alpha + x - y,$$

where α is a real-valued parameter.

a. [4 points] Find all critical points for the system, and show that for $\alpha > -1$ there are two critical points, if $\alpha = -1$ there is one, and if $\alpha < -1$ there are none.

Solution: From the second equation, at critical points we require $y = x - \alpha$. Plugging into the first, we have $0 = 3x - (x - \alpha) - x^2$, so $x^2 - 2x - \alpha = (x-1)^2 - 1 - \alpha = 0$, and $x = 1 \pm \sqrt{1+\alpha}$. If $\alpha > -1$ there are two critical points, $x = x_{1,2} = 1 \pm \sqrt{1+\alpha}$ (with $y = x_{1,2} - \alpha$); if $\alpha = -1$, there is one, $x_c = 1$ (with $y = 0$); and if $\alpha < -1$ there are no real solutions.

b. [8 points] Let $\alpha = 0$: then the system has two critical points, $(0,0)$ and $(2,2)$. Sketch a phase portrait for the nonlinear system by linearizing at critical points and determining the resulting behavior in the phase plane.

Solution: Note that the Jacobian for the system is $\mathbf{J} = \begin{pmatrix} 3-2x & -1 \\ 1 & -1 \end{pmatrix}$. If $\alpha = 0$, the critical points are $(0,0)$ and $(2,2)$. At $(0,0)$ and $(2,0)$, the Jacobians are, respectively, $\mathbf{J}(0,0) = \begin{pmatrix} 3 & -1 \\ 1 & -1 \end{pmatrix}$, and $\mathbf{J}(2,2) = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}$. The eigenvalues of the first are given by $(3-\lambda)(-1-\lambda) + 1 = \lambda^2 - 2\lambda - 2 = (\lambda - 1) - 3 = 0$, so $\lambda = 1 \pm \sqrt{3}$. Using the second row of the system, the resulting eigenvectors satisfy $v_1 + (-2 \mp \sqrt{3})v_2 = 0$, so that we may take $\mathbf{v} = \begin{pmatrix} 2 \pm \sqrt{3} \\ 1 \end{pmatrix} \approx \begin{pmatrix} 2 \pm 1.75 \\ 1 \end{pmatrix}$. Thus at $(0,0)$ we have a saddle point with outgoing trajectories along a line with slope approximately $\frac{1}{4}$, and incoming along a line with slope approximately 4.

Similarly, at $(2,2)$ eigenvalues are given by $(\lambda+1)^2 + 1 = 0$, so that $\lambda = -1 \pm i$. At $(1,0)$ we get a slope $(x,y)' = (-1,1)$, so this is an inward counter-clockwise spiral. The resulting phase portrait is shown below, with the trajectories from the linear system as solid curves and some representative nonlinear trajectories with dashed curves.