7. [14 points] The van der Pohl oscillator is a circuit that may be modeled with the system of differential equations

$$x' = -y, \qquad y' = x + (a - y^2)\, y,$$

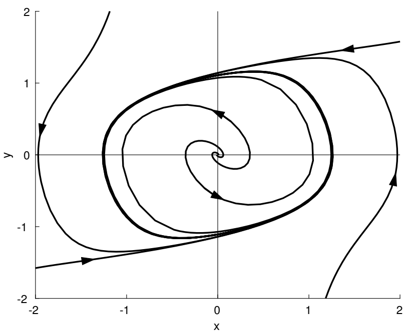
where x is the charge on a capacitor in the circuit and y is current in the circuit, scaled appropriately. The constant a is a parameter in the system.

a. [3 points] Find all critical points for this system.

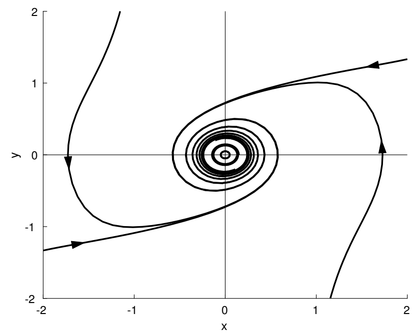
Solution: Setting $x' = y' = 0$, we have $0 = y$ (from the first equation) and $0 = x - (\alpha + x^2)y$ (from the second). With $y = 0$, the second reduces to $x = 0$. Thus the only critical point is $(x, y) = (0, 0)$.

b. [6 points] The two phase portraits (I and II) shown below are generated for the system two of the three cases $a = -1$, $a = 0$ or $a = 1$. By doing a linear analysis of the system at your critical points, determine which cases these match and explain why.

I.



II.



Solution: The Jacobian for the system is $\mathbf{J} = \begin{pmatrix} 0 & -1 \\ 1 & a - 3y^2 \end{pmatrix}$, which is at $(0, 0)$ the matrix $\mathbf{J} = \begin{pmatrix} 0 & -1 \\ 1 & a \end{pmatrix}$. The eigenvalues of this matrix are given by $-\lambda(a - \lambda) + 1 = 0$, so that $(\lambda - \frac{a}{2})^2 + 1 - a^2/4 = 0$ and $\lambda = \frac{a}{2} \pm \sqrt{\frac{a^2}{4} - 1}$. If $a = -1$, we have $\lambda = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$; if $a = 0$, $\lambda = \pm i$, and if $a = 1$, $\lambda = \frac{1}{2} \pm i\frac{\sqrt{3}}{2}$. The last ($a = 1$) corresponds to an unstable spiral source, which is what is shown in phase portrait (I). The middle case ($a = 0$) is a center, which seems characteristic of phase portrait (II).

c. [5 points] Based on your linear analysis, sketch a phase portrait for the last of the three cases $a = -1$, $a = 0$, or $a = 1$.

Solution: The remaining case is $a = -1$, a stable spiral sink. Note that if we start at $(x, y) = (1, 0)$, we have $(x', y') = (0, 1)$, so the spirals must be counter-clockwise. This leads to the phase portrait shown below.