6. [10 points] If we make a small typographical error when writing out the Lorenz system that we studied in lab 5, we obtain the system

$$\begin{aligned} x' &= \sigma(-x+y) \\ y' &= ry - x - xz \\ z' &= -bz + xy \end{aligned}$$

a. [5 points] As with the Lorenz system, one critical point of this system is $(0,0,0)$. Find a linear system that approximates the system near $(0,0,0)$.

Solution: We know that this will be $\mathbf{x}' = \mathbf{J}(0,0,0)\,\mathbf{x}$, where $\mathbf{J}$ is the Jacobian. This is

$$\mathbf{J}_0 = \begin{pmatrix} \frac{\partial}{\partial x}\sigma(-x+y) & \frac{\partial}{\partial y}\sigma(-x+y) & \frac{\partial}{\partial z}\sigma(-x+y) \\ \frac{\partial}{\partial x}(ry-x-xz) & \frac{\partial}{\partial y}(ry-x-xz) & \frac{\partial}{\partial z}(ry-x-xz) \\ \frac{\partial}{\partial x}(-bz+xy) & \frac{\partial}{\partial y}(-bz+xy) & \frac{\partial}{\partial z}(-bz+xy) \end{pmatrix}\Bigg|_{(0,0,0)}$$

$$= \begin{pmatrix} -\sigma & \sigma & 0 \\ -1-z & r & -x \\ y & x & -b \end{pmatrix}\Bigg|_{(0,0,0)} = \begin{pmatrix} -\sigma & \sigma & 0 \\ -1 & r & 0 \\ 0 & 0 & -b \end{pmatrix}.$$

And our system is, with $\mathbf{x} = \begin{pmatrix} x & y & z \end{pmatrix}^T$, $\mathbf{x}' = \mathbf{J}_0\,\mathbf{x}$.

Alternately, because we are linearizing at $(0,0,0)$ and all terms are polynomial, we can just drop the nonlinear terms from the system, obtaining the same result.

b. [5 points] If $b = 5$, $\sigma = 1$, and $r = 1/4$, the eigenvalues and eigenvectors of the coefficient matrix of the linearized system you found in (a) are approximately $\lambda_1 = -5$ and $\lambda_{2,3} = -\frac{3}{8} \pm \frac{7}{9}\,i$, with $\mathbf{v}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, and $\mathbf{v}_{2,3} = \begin{pmatrix} \frac{5}{8} \pm \frac{6}{7}\,i \\ 1 \\ 0 \end{pmatrix}$. Describe phase space trajectories in this case. If we start with an initial condition $(x,y,z) = (0.5, 0.5, 0)$, sketch the trajectory in the phase space.

Solution: We note that the first eigenvalue is a much larger (in magnitude) negative real value than the real part of the other two; thus, we expect that to decay much faster than the other. Because this is associated with the eigenvector $\mathbf{v}_1 = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}^T$, this means that the trajectories will rapidly decay into the xy-plane. Once there, they will be spiral trajectories. We note that the linearized system gives $\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} -1 & 1 \\ -1 & 1/4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$, so starting from $(x,y) = (1,1)$, $(x', y') = (0, -3/4)$ and the trajectory must be moving clockwise. Thus we have a clockwise spiral in the xy-plane, shown below.