6. [10 points] If we make a small typographical error when writing out the Lorenz system that we studied in lab 5, we obtain the system
\[
\begin{align*}
x' &= \sigma(-x + y) \\
y' &= ry - x - xz \\
z' &= -bz + xy
\end{align*}
\]

a. [5 points] As with the Lorenz system, one critical point of this system is (0, 0, 0). Find a linear system that approximates the system near (0, 0, 0).

**Solution:** We know that this will be \( x' = J(0, 0, 0) x \), where \( J \) is the Jacobian. This is
\[
J_0 = \begin{pmatrix}
\frac{\partial}{\partial x} \sigma(-x + y) & \frac{\partial}{\partial y} \sigma(-x + y) & \frac{\partial}{\partial z} \sigma(-x + y) \\
\frac{\partial}{\partial x} (ry - x - xz) & \frac{\partial}{\partial y} (ry - x - xz) & \frac{\partial}{\partial z} (ry - x - xz) \\
\frac{\partial}{\partial x} (-bz + xy) & \frac{\partial}{\partial y} (-bz + xy) & \frac{\partial}{\partial z} (-bz + xy)
\end{pmatrix}
\]
\[J(0,0,0) = \begin{pmatrix}
-\sigma & \sigma & 0 \\
-1 & 1 & -x \\
y & x & -b
\end{pmatrix}
\]

And our system is, with \( x = (x, y, z)^T \), \( x' = J_0 x \).

Alternately, because we are linearizing at (0, 0, 0) and all terms are polynomial, we can just drop the nonlinear terms from the system, obtaining the same result.

b. [5 points] If \( b = 5 \), \( \sigma = 1 \), and \( r = 1/4 \), the eigenvalues and eigenvectors of the coefficient matrix of the linearized system you found in (a) are approximately \( \lambda_1 = -5 \) and \( \lambda_{2,3} = -\frac{3}{8} \pm \frac{7}{9} i \), with \( v_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \), and \( v_{2,3} = \begin{pmatrix} \frac{5}{8} \pm \frac{7}{9} i \\ 1 \\ 0 \end{pmatrix} \). Describe phase space trajectories in this case. If we start with an initial condition \((x, y, z) = (0.5, 0.5, 0)\), sketch the trajectory in the phase space.

**Solution:** We note that the first eigenvalue is a much larger (in magnitude) negative real value than the real part of the other two; thus, we expect that to decay much faster than the other. Because this is associated with the eigenvector \( v_1 = (0, 0, 1)^T \), this means that the trajectories will rapidly decay into the \( xy \)-plane. Once there, they will be spiral trajectories. We note that the linearized system gives \( \begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} -1 & 1 \\ -1 & 1/4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \), so starting from \((x, y) = (1, 1)\), \((x', y') = (0, -3/4)\) and the trajectory must be moving clockwise. Thus we have a clockwise spiral in the \( xy \)-plane, shown below.