7. [15 points] Consider a mass-spring system with a nonlinear "soft" spring, for which the displacement x of a mass attached to the spring is modeled by

$$x'' + 2\gamma_0 x' + k(x - x^2) = 0.$$

a. [4 points] Rewrite this as a system in $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ x' \end{pmatrix}$.

Solution: We have

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} y \\ k(x^2 - x) - 2\gamma_0 y \end{pmatrix}.$$

b. [5 points] Find all critical points for your system from (a).

Solution: To find critical points we require that $x' = y' = 0$. Thus $y = 0$, and $x^2 - x = 0$, so $x = 0$ or $x = 1$. The two critical points are $(0, 0)$ and $(1, 0)$.

Problem 7, continued.
We are solving

$$x'' + 2\gamma_0 x' + k(x - x^2) = 0.$$

You may want to write your system from (a) here:

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} y \\ k(x^2 - x) - 2\gamma_0 y \end{pmatrix}.$$

c. [6 points] Let $\gamma_0 = 4$ and $k = 18$. Sketch the phase plane for the system in this case by linearizing about all critical points and determining local behavior. Using your sketch, what do you expect to happen to a solution that starts with the initial condition $x(0) = 0.8$, $x'(0) = y(0) = 0.2$? *(Note: for this part of the problem you should assume that the original equation is in fact well-defined for $x < 0$.)*

Solution: The Jacobian of the system is $J = \begin{pmatrix} 0 & 1 \\ 18(2x-1) & -8 \end{pmatrix}$. At $(0,0)$, we have $J_0 = \begin{pmatrix} 0 & 1 \\ -18 & -8 \end{pmatrix}$. Eigenvalues of J_0 satisfy $\lambda^2 + 8\lambda + 18 = 0$, or $(\lambda+4)^2 + 2 = 0$, so $\lambda = -4 \pm i\sqrt{2}$. At $(1,0)$, $\begin{pmatrix} x \\ y \end{pmatrix}' = J_0 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -18 \end{pmatrix}$. This is downward, so near $(0,0)$ we must have a clockwise spiral sink.

At $(1,0)$, the Jacobian is $J_1 = \begin{pmatrix} 0 & 1 \\ 18 & -8 \end{pmatrix}$. Eigenvalues satisfy $\lambda^2 + 8\lambda - 18 = (\lambda+4)^2 - 34 = 0$, so $\lambda = -4 \pm \sqrt{34}$. Note that $\sqrt{34} \approx \sqrt{36} = 6$, so these are approximately $\lambda_{1,2} = -10, 2$. Then eigenvectors satisfy $(4 \mp \sqrt{34})v_1 + v_2 = 0$, so the eigenvectors are $\mathbf{v}_{1,2} = \begin{pmatrix} 1 \\ -4 \pm \sqrt{34} \end{pmatrix} \approx \begin{pmatrix} 1 \\ -10 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. This is a(n unstable) saddle point. Sketching these in the phase plane, we obtain the graph shown below.

Then, if we start at $(0.8, 0.2)$, we are to the left of the critical point $(1,0)$ and below the attracting eigenline. We therefore expect the trajectory to move out and down toward $(1,0)$, crossing the x-axis to the left of $(1,0)$ and then spiraling in to $(0,0)$. The phase portrait is shown below, with the trajectory starting at $(0.8, 0.2)$ shown with an initial dot and thicker lines. The dashed curves are nonlinear trajectories.