8. [15 points] For each of the following, identify the statement as true or false by circling "True" or "False" as appropriate, and provide a short (one or two sentence) explanation indicating why that answer is correct.

a. [3 points] Two linearly independent solutions of $x'' + 6x' + 9x = 0$ are $x_1 = e^{-3t}$ and $x_2 = te^{-3t}$. Thus two linearly independent solutions of $\mathbf{x}' = \begin{pmatrix} 0 & 1 \\ -9 & -6 \end{pmatrix} \mathbf{x}$ are $\mathbf{x}_1 = \begin{pmatrix} 1 \\ -3 \end{pmatrix} e^{-3t}$ and $\mathbf{x}_2 = \begin{pmatrix} 1 \\ -3 \end{pmatrix} te^{-3t}$.

True [False]

Solution: There are a number of ways to see that this is false: first is that $\mathbf{x}_2$ isn't a solution to the system ($\mathbf{x}_2' = \begin{pmatrix} -3t+1 \\ 9t-3 \end{pmatrix} e^{-3t}$, while the right hand side is $\begin{pmatrix} -3t \\ 9t \end{pmatrix} e^{-3t}$.) In addition, $\mathbf{x}_1$ and $\mathbf{x}_2$ aren't linearly independent ($W(\mathbf{x}_1, \mathbf{x}_2) = 0$).

b. [3 points] If $\mathbf{A}$ is a real-valued 5×5 matrix with 5 distinct eigenvalues, not necessarily real, and if the real parts of all of the eigenvalues are negative, then $\mathbf{x} = \mathbf{0}$ is an asymptotically stable critical point of $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

[True] False

Solution: All terms in the general solution to $\mathbf{x}' = \mathbf{A}\mathbf{x}$ will be multiplied by a factor of $e^{\mathrm{Re}(\lambda_j)t}$ $(1 \le j \le 5)$, with the only other functional dependence being $\cos(\omega t)$, $\sin(\omega t)$ or powers of t. Thus the solutions all decay to zero, and the critical point is stable.

c. [3 points] If the nonlinear system $\mathbf{x}' = \mathbf{f}(\mathbf{x})$ has an unstable isolated critical point $\mathbf{x} = \mathbf{x}_0$, then any solution to the system will eventually get infinitely far from $\mathbf{x}_0$.

True [False]

Solution: There are lots of counterexamples. A saddle point has a line of solutions that approach the critical point asymptotically. Another example is the van der Pol system: for certain parameter values $\mathbf{x} = \mathbf{0}$ is an unstable critical point, but all solutions are attracted to a limit cycle which does not allow trajectories to escape to infinity.

d. [3 points] Suppose that the nonlinear system $x' = F(x, y)$, $y' = G(x, y)$ has an isolated critical point $(x, y) = (1, 2)$. If we are able to linearize the system at this critical point and the eigenvalues of the resulting coefficient matrix are real-valued and non-zero, we can deduce the stability of the critical point from the linearization.

[True] False

Solution: This is the substance of our theorem about the linear analysis of almost linear systems; if the eigenvalues are real and non-zero, small changes will not change our conclusion as to the stability of the point. If the eigenvalues are equal we may not be able to determine if the point is a node or a spiral point, but its stability will remain the same.

e. [3 points] $\mathcal{L}^{-1}\{\frac{e^{-2s}s}{(s+1)^2+4}\} = e^{-(t-2)} \cos(2(t-2))u_2(t)$

True [False]

Solution: To use the rule for $F(s-c)$, all terms with s must first be rewritten as $s-c$: $\mathcal{L}^{-1}\{\frac{e^{-2s}s}{(s+1)^2+4}\} = \mathcal{L}^{-1}\{\frac{e^{-2s}((s+1)-1)}{(s+1)^2+4}\} = e^{-(t-2)}(\cos(2(t-2)) - \frac{1}{2}\sin(2(t-2)))u_2(t)$.