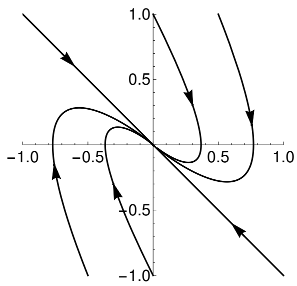
5. [12 points] Each of the following requires a short (one equation or formula) answer. Provide the required answer, and a short (one or two sentence) explanation.

a. [3 points] Write a linear, constant coefficient, second order, nonhomogeneous differential equation for which the method of undetermined coefficients is not applicable.

Solution: We must have a forcing term that is not of polynomial, exponential, or sinusoidal function, or a product of those. One such example is

$$y'' + y = \tan(t).$$

b. [3 points] Write a linear, constant coefficient, second order differential equation that has the phase portrait shown to the right.



Solution: The phase portrait shows a repeated eigenvalue (root of the characteristic equation), so this should be something like

$$y'' + 2y' + y = 0.$$

This has the solutions $y_1 = e^{-t}$ and $y_2 = te^{-t}$, so the eigenvector is $\begin{pmatrix}1 & -1\end{pmatrix}^T$ $(= \begin{pmatrix}y_1(0) & y_1'(0)\end{pmatrix}^T)$, as shown. (Note also that the second solution is $\mathbf{x} = (t\mathbf{v} + \begin{pmatrix}0 & 1\end{pmatrix}^T)e^{-t}$, which completes the phase portrait shown.)

c. [3 points] If $L[y] = f(t)$ is a linear, constant coefficient, second order differential equation and $L[y] = 0$ is solved by $y = c_1e^{-t} + c_2te^{-t}$, write a function $f(t)$ for which a good solution guess would be $y = At^3e^{-t} + Bt^2e^{-t}$.

Solution: Note that if $f(t) = e^{-t}$, we must guess $y = At^2e^{-t}$ to avoid having any term in the guess that is part of the homogeneous solution. Therefore we can take

$$f(t) = kte^{-t},$$

for any k; our guess is then $y = t^2(Ate^{-t} + Be^{-t})$, as desired.

d. [3 points] Write a linear, constant coefficient, second order differential equation having a phase portrait that is a spiral sink converging on the point $(2, 0)$.

Solution: For the phase portrait to show a spiral sink the characteristic equation must have complex conjugate roots with negative real part. One such is $r^2 + 2r + 2 = 0$ (for which $r = -1 \pm i$), so that the linear operator is $L = D^2 + 2D + 2$. Then, for the critical point to be $(2, 0)$, we must have an equilibrium solution $x = 2$. The equation could therefore be

$$x'' + 2x' + 2x = 4.$$