7. [16 points] Our model for a ruby laser is, with N = the population inversion of atoms and P = the intensity of the laser,

$$N' = \gamma A - \gamma N(1+P), \quad P' = P(N-1).$$

In lab we found that the critical points of this system are $(N,P) = (A,0)$ and $(N,P) = (1, A-1)$. For this problem we will assume that $\gamma = \frac{1}{2}$; A is, of course, also a constant.

a. [4 points] Find a linear system that approximates the nonlinear system near the critical point $(A,0)$. Show that if $A < 1$ this critical point is asymptotically stable, and if $A > 1$ it is unstable.

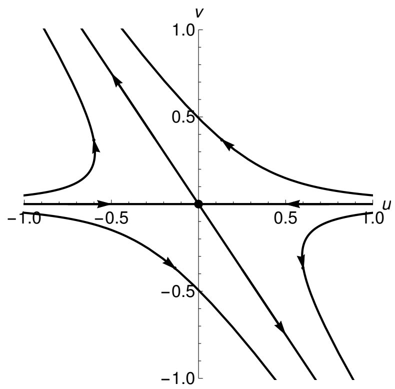
Solution: The easiest way to find the linearization is to use the Jacobian. Here we have $N' = F(N,P) = \frac{A}{2} - \frac{1}{2}(N + NP)$, $P' = G(N,P) = PN - P$, so that

$$J = \begin{pmatrix} F_N & F_P \\ G_N & G_P \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}(1+P) & -\frac{1}{2}N \\ P & N-1 \end{pmatrix}.$$

At $(A,0)$, this is $J(A,0) = \begin{pmatrix} -\frac{1}{2} & -\frac{A}{2} \\ 0 & A-1 \end{pmatrix}$, which has eigenvalues $\lambda = -\frac{1}{2}$ and $\lambda = A-1$. Thus, if $A < 1$ both eigenvalues are real and negative, and the critical point is asymptotically stable; if $A > 1$, the second eigenvalue is positive and the critical point becomes unstable.

b. [6 points] Suppose that the linear system you obtained in (a) is, for some value of A, $u' = -\frac{1}{2}u - v$, $v' = v$. Sketch a phase portrait that shows solution trajectories of the linear system. Explain how these trajectories are related to trajectories in the (N,P) phase plane.

Solution: Note that this is $\mathbf{u}' = \begin{pmatrix} -\frac{1}{2} & -1 \\ 0 & 1 \end{pmatrix}\mathbf{u}$, so this is apparently the result we obtained in (a) with $A = 2$. Eigenvalues of the coefficient matrix are $\lambda = -\frac{1}{2}$ and $\lambda = 1$. When $\lambda = -\frac{1}{2}$ we have the eigenvector $\mathbf{v} = \begin{pmatrix} 1 & 0 \end{pmatrix}^T$, and when $\lambda = 1$, $\mathbf{v} = \begin{pmatrix} -2 & 3 \end{pmatrix}^T$. These give the saddle point shown below.



These trajectories will be very similar to the trajectories in the (N,P) plane at the critical point, $(A,0)$.

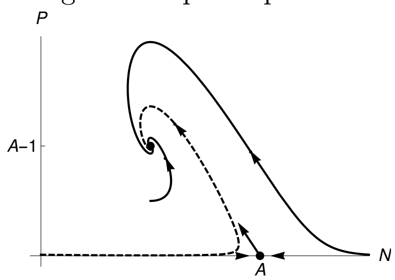
Problem 7, cont. We are considering the system

$$N' = \frac{1}{2}A - \frac{1}{2}N(1+P), \quad P' = P(N-1),$$

which has critical points $(N, P) = (A, 0)$ *and* $(N, P) = (1, A-1)$.

c. [6 points] Suppose that, for the value of A used in (b), the coefficient matrix for the linear system approximating (N, P) near the critical point $(1, A-1)$ is $\begin{pmatrix} -1 & -\frac{1}{2} \\ 1 & 0 \end{pmatrix}$, which has eigenvalues $\lambda = \frac{1}{2}(-1 \pm i)$. Using this information with your work in (b), sketch a representative solution curve for P as a function of t, if $P(0) = 0.01$ when $N(0) = 0$.

Solution: We note that in the phase plane for the nonlinear system, critical points are at $(A, 0)$ and $(1, A-1)$. The phase portrait near the former is given in (b); for the latter, we know it is a spiral sink, and, because $\begin{pmatrix} -1 & -\frac{1}{2} \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$, the inward spiral must be counter clockwise. This gives the phase portrait shown below.



A trajectory starting at $(0, 0.01)$ is suggested by the dashed curve. Reading the behavior of P from this, we get the curve below. We know that it starts at $(0, 0.01)$, that it must remain close to the t-axis for a while, and then must oscillate around and converge to the line $P = A - 1$. Finally, note that we do not know the time scale on which these transitions take.