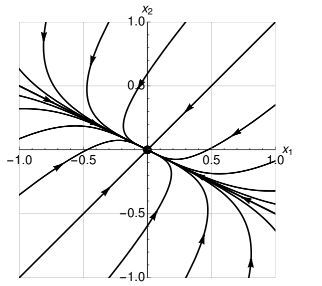
1. [12 points] Six matrices and their eigenvalues and eigenvectors are given below. Use this information to answer the questions below. Be sure that you explain your answers.

$\mathbf{A}_1$	$\mathbf{A}_2$	$\mathbf{A}_3$	$\mathbf{A}_4$	$\mathbf{A}_5$	$\mathbf{A}_6$
$\begin{pmatrix}-1 & 2\\ -1 & -3\end{pmatrix}$	$\begin{pmatrix}2 & 2\\ 1 & 3\end{pmatrix}$	$\begin{pmatrix}-2 & 2\\ 1 & -3\end{pmatrix}$	$\begin{pmatrix}-1 & 3\\ 2 & -2\end{pmatrix}$	$\begin{pmatrix}-2 & -2\\ -1 & -3\end{pmatrix}$	$\begin{pmatrix}-3 & -1\\ 1 & -1\end{pmatrix}$
$\lambda_{1,2} = -2 \pm i$	$\lambda_{1,2} = 1, 4$	$\lambda_{1,2} = -4, -1$	$\lambda_{1,2} = -4, 1$	$\lambda_{1,2} = -4, -1$	$\lambda_{1,2} = -2, -2$
$\mathbf{v}_1 = \begin{pmatrix}2\\ -1+i\end{pmatrix}$	$\mathbf{v}_1 = \begin{pmatrix}-2\\ 1\end{pmatrix}$	$\mathbf{v}_1 = \begin{pmatrix}-1\\ 1\end{pmatrix}$	$\mathbf{v}_1 = \begin{pmatrix}-1\\ 1\end{pmatrix}$	$\mathbf{v}_1 = \begin{pmatrix}1\\ 1\end{pmatrix}$	$\mathbf{v}_1 = \begin{pmatrix}-1\\ 1\end{pmatrix}$
$\mathbf{v}_2 = \begin{pmatrix}2\\ -1-i\end{pmatrix}$	$\mathbf{v}_2 = \begin{pmatrix}1\\ 1\end{pmatrix}$	$\mathbf{v}_2 = \begin{pmatrix}2\\ 1\end{pmatrix}$	$\mathbf{v}_2 = \begin{pmatrix}3\\ 2\end{pmatrix}$	$\mathbf{v}_2 = \begin{pmatrix}-2\\ 1\end{pmatrix}$	$\mathbf{w} = \begin{pmatrix}1\\ 0\end{pmatrix}$

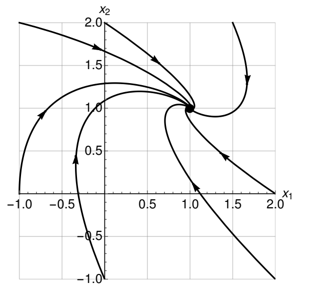
a. [6 points] Write a linear system involving one of the $\mathbf{A}_j$ that could have the phase portrait shown to the right.



Solution: We note that there is a critical point at the origin, so we are solving a homogeneous system $\mathbf{x}' = \mathbf{A}\mathbf{x}$. All solutions approach the origin, and there are two straight-line solutions, so there must be two real negative eigenvalues. This means our coefficient matrix is either $\mathbf{A}_3$ or $\mathbf{A}_5$. Then we note that the eigenvalue associated with an eigenvector with negative slope is larger, because all trajectories not on an eigenvector asymptotically approach a line in the direction of that eigenvector. This is the case for $\mathbf{A}_5$, where $\lambda = -4$ and $\lambda = -1$, and the eigenvector associated with $\lambda = -1$ has negative slope—and the eigenvectors correspond to the lines $y = x$ and $y = x/2$ seen in the phase portrait. Thus our system is

$$\mathbf{x}' = \begin{pmatrix}-2 & -2\\ -1 & -3\end{pmatrix}\mathbf{x}.$$

b. [6 points] Write a linear system involving one of the $\mathbf{A}_j$ that could have the phase portrait shown to the right.



Solution: Note that the critical point is in this case at $(1, 1)$, not the origin. Thus our system must be non-homogeneous, $\mathbf{x}' = \mathbf{A} + \mathbf{k}$. There are no straight-line trajectories shown in the phase portrait, so the eigenvalues of $\mathbf{A}$ must be complex, which is the case for $\mathbf{A}_1$; further, $\mathbf{A}_1$ predicts that above the critical point slopes will be $\begin{pmatrix}x_1'\\ x_2'\end{pmatrix} = \mathbf{A}_1\begin{pmatrix}0\\ 1\end{pmatrix} = \begin{pmatrix}2\\ -3\end{pmatrix}$, which is consistent with the phase portrait. Finally, for the critical point to be at $(1, 1)$, we must have $\mathbf{0} = \begin{pmatrix}-1 & 2\\ -1 & -3\end{pmatrix}\begin{pmatrix}1\\ 1\end{pmatrix} + \mathbf{k}$, so that $\mathbf{k} = \begin{pmatrix}-1\\ 4\end{pmatrix}$, and our system is

$$\mathbf{x}' = \begin{pmatrix}-1 & 2\\ -1 & -3\end{pmatrix}\mathbf{x} + \begin{pmatrix}-1\\ 4\end{pmatrix}.$$