1. [12 points] Six matrices and their eigenvalues and eigenvectors are given below. Use this information to answer the questions below. Be sure that you explain your answers.

<table>
<thead>
<tr>
<th>$A_1$</th>
<th>$A_2$</th>
<th>$A_3$</th>
<th>$A_4$</th>
<th>$A_5$</th>
<th>$A_6$</th>
</tr>
</thead>
</table>
| \[
\begin{pmatrix}
-1 & 2 \\
-1 & -3
\end{pmatrix}
\] | \[
\begin{pmatrix}
2 & 2 \\
1 & 3
\end{pmatrix}
\] | \[
\begin{pmatrix}
-2 & 2 \\
1 & -3
\end{pmatrix}
\] | \[
\begin{pmatrix}
-1 & 3 \\
2 & -2
\end{pmatrix}
\] | \[
\begin{pmatrix}
-2 & -2 \\
-1 & -3
\end{pmatrix}
\] | \[
\begin{pmatrix}
-3 & -1 \\
1 & -1
\end{pmatrix}
\] |
| $\lambda_{1,2} = -2 \pm i$ | $\lambda_{1,2} = 1, 4$ | $\lambda_{1,2} = -4, -1$ | $\lambda_{1,2} = -4, 1$ | $\lambda_{1,2} = -4, -1$ | $\lambda_{1,2} = -2, -2$ |
| $v_1 = \begin{pmatrix} 2 \\ -1 + i \end{pmatrix}$ | $v_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$ | $v_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ | $v_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ | $v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ | $v_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ |
| $v_2 = \begin{pmatrix} 2 \\ -1 - i \end{pmatrix}$ | $v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ | $v_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ | $v_2 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ | $v_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$ | $w = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ |

a. [6 points] Write a linear system involving one of the $A_j$ that could have the phase portrait shown to the right.

Solution: We note that there is a critical point at the origin, so we are solving a homogeneous system $x' = Ax$. All solutions approach the origin, and there are two straight-line solutions, so there must be two real negative eigenvalues. This means our coefficient matrix is either $A_3$ or $A_5$. Then we note that the eigenvalue associated with an eigenvector with negative slope is larger, because all trajectories not on an eigenvector asymptotically approach a line in the direction of that eigenvector. This is the case for $A_5$, where $\lambda = -4$ and $\lambda = -1$, and the eigenvector associated with $\lambda = -1$ has negative slope—and the eigenvectors correspond to the lines $y = x$ and $y = x/2$ seen in the phase portrait. Thus our system is

$$x' = \begin{pmatrix} -2 & -2 \\ -1 & -3 \end{pmatrix} x.$$

b. [6 points] Write a linear system involving one of the $A_j$ that could have the phase portrait shown to the right.

Solution: Note that the critical point is in this case at $(1, 1)$, not the origin. Thus our system must be non-homogeneous, $x' = A + k$. There are no straight-line trajectories shown in the phase portrait, so the eigenvalues of $A$ must be complex, which is the case for $A_1$; further, $A_1$ predicts that above the critical point slopes will be $\begin{pmatrix} x'_1 \\ x'_2 \end{pmatrix} = A_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$, which is consistent with the phase portrait. Finally, for the critical point to be at $(1, 1)$, we must have $0 = \begin{pmatrix} -1 & 2 \\ -1 & -3 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + k$, so that $k = \begin{pmatrix} -1 \\ 4 \end{pmatrix}$, and our system is

$$x' = \begin{pmatrix} -1 & 2 \\ -1 & -3 \end{pmatrix} x + \begin{pmatrix} -1 \\ 4 \end{pmatrix}.$$