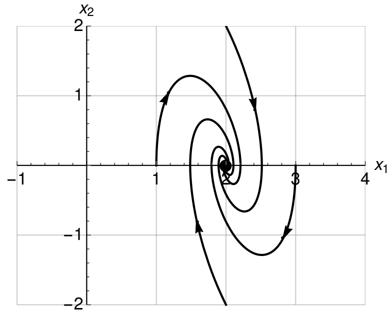
3. [12 points] Suppose a model for a physical system (e.g., a circuit or a mass-spring system) is given by the differential equation $L[y] = y'' + ay' + by = k$ (where a, b, and k are real numbers).

a. [4 points] If the solution to the problem with some initial conditions is $y = e^{-t}\cos(2t) - e^{-t}\sin(2t) + 2$, what can you say about a, b, and k?

Solution: We see that eigenvalues are $\lambda = -1 \pm 2i$, so the characteristic polynomial is $p(\lambda) = (\lambda + 1)^2 + 4 = \lambda^2 + 2\lambda + 5$ and $a = 2$, $b = 5$. Then $y_p = 2$, so $k = 10$.

b. [4 points] If the solution to the problem with some initial conditions is $y = e^{-t}\cos(2t) - e^{-t}\sin(2t) + 2$, sketch a phase portrait for the system. Be sure it is clear how you obtain your solution.

Solution: This is a spiral sink centered on $(2, 0)$. Note that $y(0) = 3$, and as t increases y looks like $(\cos(2t), -\sin(2t))$, so the spiral is clockwise. This can also be seen by writing the equation as a system and checking the direction of motion.



c. [4 points] Now suppose that the solution to the problem with some initial conditions is $y = e^{-t}\cos(2t) - e^{-t}\sin(2t) + 2$, and that at some time $t = t_0$ we remove the forcing term (k). Write a single differential equation you could solve to find y for all $t \geq 0$. What initial conditions apply at $t = 0$?

Solution: The equation is $y'' + 2y' + 5y = 10(1 - u_{t_0}(t))$; we can see the initial conditions from the solution: $y(0) = 3$, $y'(0) = -3$.