4. [12 points] Consider the predator-prey model with harvesting (harvesting here implies hunting by humans, e.g., fishing if the populations are fish) given by

$$x' = x\,(3 - x - y) - 2, \quad y' = y\,(-3 + x)\,.$$

Note that as x and y are populations, we must have $x, y \geq 0$.

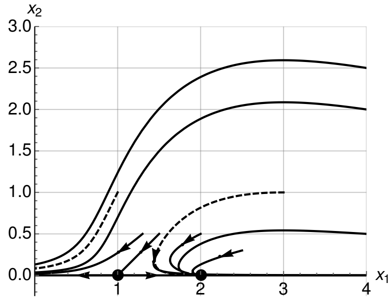
a. [3 points] Explain what each term in the equation for x models. Is x or y the predator? Which population is being harvested?

Solution: In the equation for x, the term $3x$ is a birth/death rate term; $-x^2$ is a logistic resource limitation term; $-xy$ is the species interaction term, and as it is negative we know x must be the prey; and the -2 must be the harvesting term, so x is also being harvested.

b. [7 points] By doing an appropriate linear analysis, sketch a phase portrait for this system.

Solution: First we find critical points: if $y' = 0$, we need $y = 0$ or $x = 3$. Then, if $x' = 0$ and $x = 3$, we have $-3y - 2 = 0$, so that $y = -\frac{2}{3}$. This doesn't make sense for our model, so we ignore it. If $y = 0$, $x' = 0$ requires $-x^2 + 3x - 2 = -(x-2)(x-1) = 0$, so $x = 1$ or $x = 2$. The physically relevant critical points are therefore $(1, 0)$ and $(2, 0)$.

Then, the Jacobian for the system is $\mathbf{J} = \begin{pmatrix} 3 - 2x - y & -x \\ y & -3 + x \end{pmatrix}$.



At $(1, 0)$, $\mathbf{J}(1,0) = \begin{pmatrix} 1 & -1 \\ 0 & -2 \end{pmatrix}$, so that eigenvalues of the linearized system are $\lambda = 1, -2$. Corresponding eigenvectors are $\mathbf{v} = \begin{pmatrix} 1 & 0 \end{pmatrix}^T$ and $\mathbf{v} = \begin{pmatrix} 1 & 1 \end{pmatrix}^T$. At $(2, 0)$, $\mathbf{J}(2,0) = \begin{pmatrix} -1 & -2 \\ 0 & -1 \end{pmatrix}$, so $\lambda = -1$ with $\mathbf{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. We could do the analysis without the generalized eigenvector, but it satisfies $\begin{pmatrix} 0 & -2 \\ 0 & 0 \end{pmatrix} \mathbf{w} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, so that $\mathbf{w} = \begin{pmatrix} 0 & -\frac{1}{2} \end{pmatrix}^T$. Note that this means that a trajectory starting immediately below the critical point will initially move to the right.

Finally, note that the y-nullclines ($y' = 0$) are $y = 0$ and $x = -3$; the x-nullcline is harder to visualize, but is given by $y = 3 - x - \frac{2}{x}$. Putting these together, we get the phase portrait shown to the right, above.

c. [2 points] Based on your answer to (b), sketch what you expect the behavior of the solution to the system will be as a function of time if $x(0) = 3$ and $y(0) = 1$. How would you expect this to differ from the behavior with the initial condition $x(0) = 1$, $y(0) = 1$?

Solution: Starting at $(3, 1)$ the trajectory should move to the right and down, overshooting the critical point at $(2, 0)$ but then coming back to converge to it. From $(1, 1)$, we move to the left and down until $x = 0$. At that point population x vanishes and $y' = -3y$, so y also decays to zero. These are shown by the two dashed curves above. The corresponding trajectories are given below.