5. [10 points] Consider the linear system

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}' = \begin{pmatrix} -1 & 0 & \alpha^2 \\ 0 & -2 & 2 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

a. [5 points] For what values of α, if any, will all solutions to the system remain bounded as $t \to \infty$?[1]

Solution: Solutions will be unbounded if there are any eigenvalues λ of the coefficient matrix for which $\mathrm{Re}(\lambda) > 0$, so we find the eigenvalues of the matrix. Eigenvalues satisfy $\det(\mathbf{A} - \lambda\mathbf{I}) = (-1-\lambda)((-2-\lambda)(-1-\lambda)-0)-0+\alpha^2(0-(-2-\lambda)) = (-2-\lambda)((\lambda+1)^2 - \alpha^2) = 0$. Thus $\lambda = -2$ or $\lambda = -1 \pm \alpha$, and all solutions will remain bounded provided $|\alpha| \leq 1$ (that is, $-1 \leq \alpha \leq 1$).

b. [5 points] Now suppose that $\alpha = 2$. Are there any initial conditions for which solutions to the system will remain bounded? If so, what are they? Explain.

Solution: From (a), the eigenvalues are $\lambda = -2$, $\lambda = -3$, and $\lambda = 1$. Thus we will have bounded solutions if we start with an initial condition that includes of only the solutions associated with the first two eigenvalues. To see what these are, we first find the eigenvectors. For the three eigenvalues we have: for $\lambda = -3$, $\mathbf{A} - \lambda\mathbf{I} = \begin{pmatrix} 2 & 0 & 4 \\ 0 & 1 & 2 \\ 1 & 0 & 2 \end{pmatrix}$; for $\lambda = -2$, $\mathbf{A} - \lambda\mathbf{I} = \begin{pmatrix} 1 & 0 & 4 \\ 0 & 0 & 2 \\ 1 & 0 & 1 \end{pmatrix}$; and for $\lambda = 1$, $\mathbf{A} - \lambda\mathbf{I} = \begin{pmatrix} -2 & 0 & 4 \\ 0 & -3 & 2 \\ 1 & 0 & -2 \end{pmatrix}$. Thus for $\lambda = -3$, $v_1 = -2v_3$ and $v_2 = -2v_3$, and we can take $\mathbf{v} = \begin{pmatrix} 2 & 2 & -1 \end{pmatrix}^T$. For $\lambda = -2$, we can take $\mathbf{v} = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}^T$. Finally, for $\lambda = 1$, we have $v_1 = 2v_3$ and $3v_2 = 2v_3$, and can take $\mathbf{v} = \begin{pmatrix} 6 & 2 & 3 \end{pmatrix}$. The general solution to the problem is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = c_1 \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix} e^{-3t} + c_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} e^{-2t} + c_3 \begin{pmatrix} 6 \\ 2 \\ 3 \end{pmatrix} e^{t}.$$

Any initial condition that is a linear combination of the first two eigenvectors will remain bounded as $t \to \infty$. (This is the plane $x = 2u$, $y = 2u + v$, $z = -u$, for $u, v \in \mathbb{R}$.)

[1] *Possibly useful:* $\det\left(\begin{pmatrix} a & 0 & b \\ 0 & c & d \\ e & 0 & f \end{pmatrix}\right) = acf - bce.$