

Math 115 – Global Extrema Problems

Optimization, i.e. finding the global minimum and/or global maximum values of a function, is a very important application of calculus.

As you read these examples, take note of the highlighted **required** key features in each solution. Notice that the work required when finding **global** extrema is typically quite different from the work required when finding **local** extrema.

Example 1

Let

$$f(x) = \begin{cases} \frac{2-x}{e} & x < 1 \\ x^2 e^{-x} & x \geq 1. \end{cases}$$

Find the x -coordinate(s) of all global extrema of $f(x)$ on $[-3, 3]$. You must use calculus to find your answers, and be sure to show enough evidence to fully justify your answers.

check continuity f is continuous, since $\frac{2-1}{e} = \frac{1}{e}$ and $1^2 e^{-1} = \frac{1}{e}$

check differentiability $f'(x) = \begin{cases} -\frac{1}{e} & x < 1 \\ x^2(-e^{-x}) + 2xe^{-x} & x > 1 \end{cases}$

$= \begin{cases} -\frac{1}{e} & x < 1 \\ xe^{-x}(2-x) & x > 1 \end{cases}$

$\underline{f'(x) \text{ DNE at } x=1}$ since $1 \cdot e^{-1}(2-1) = \frac{1}{e} \neq -\frac{1}{e}$

find critical points algebraically $\underline{f'(x) = 0: -\frac{1}{e} \text{? never zero}}$

$xe^{-x}(2-x) = 0$ when $x=0$ or 2

$\hookrightarrow$ not in domain of this piece

So $x=1, 2$ are the critical points

x	$f(x)$
-3	$\frac{5}{e} \approx 1.839$
1	$\frac{1}{e} \approx 0.368$
2	$\frac{4}{e^2} \approx 0.541$
3	$\frac{9}{e^3} \approx 0.448$

consider values of f at critical points and endpoints of interval

By EVT, f has a global max at $x=-3$ and a global min at $x=1$.

Note: Calculators are no longer allowed on exams, so exams will be written accordingly. Do note that since e is a positive number, one can tell that e.g. $5/e$ is greater than $1/e$. Similarly, since e is approximately 2.7, one can tell that e.g. $1/e$ is less than 1. (The other values here are more difficult to judge.)

- Don't forget to check for continuity and differentiability, particularly for piecewise functions.
- Find critical points algebraically. **Don't forget** to check for points where f' does not exist.
- Consider the value of f at all relevant critical points.
- Consider the value of f at the endpoints of the interval.

Example 2a

A function $g(x)$ and its derivative $g'(x)$ are given below.

$$g(x) = \frac{(x-1)^2}{x^3}$$

$$g'(x) = -\frac{(x-1)(x-3)}{x^4}$$

Find the x -coordinate(s) of all global extrema of $g(x)$ on $(0, \infty)$. You must use calculus to find your answers, and be sure to show enough evidence to fully justify your answers.

$$g'(x) = 0: x = 1, 3$$

$$g'(x) \text{ DNE } x=0, \text{ not in domain}$$

consider values
of g at critical
points

$$g(1) = 0$$

$$g(3) = \frac{4}{27}$$

$$\lim_{x \rightarrow 0^+} g(x) = \infty$$

$$\lim_{x \rightarrow \infty} g(x) = 0$$

consider
end behavior

so g has a global min at $x=1$
and no global max

- Consider the value of g at all relevant critical points.
- Consider the behavior of g at each end of the interval.

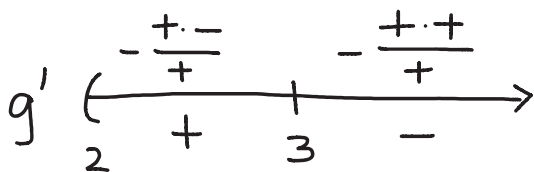
Example 2b

Suppose we had been asked to find the location(s) of all global extrema of $g(x)$ on $(2, \infty)$ instead. We could certainly provide a solution similar to that above. However, because in this case there is **only one** critical point of $g(x)$ in the interval $(2, \infty)$, we can provide the following alternate solution.

$$g'(x) = 0: x = 1, 3, \text{ only } 3 \text{ is in } (2, \infty)$$

$$g'(x) \text{ DNE } x=0, \text{ not in domain}$$

justify
whether
the one
critical point
is a local
extrema



by 1st Deriv. Test,
local max at $x=3$

Since g is continuous on $(2, \infty)$ and there is only one critical point on this open interval, the local max at $x=3$ must also be the the global max, and because there no other candidates for global min, there is no global min.

clear conclusion

(Note that if this interval had one or more endpoints included, we would need to analyze those as well.)

- Show that the one critical point in the interval is a local max or min, including all necessary justification (see Local Extrema and Inflection Point Problems).
- Clearly state that there is only one critical point in the interval when justifying the conclusion.