

Section 2.4: Interpretations of the Derivative

This section explores the meaning of the derivative of a function when considered in a particular real-world context. This topic is a very important piece of Michigan's Calculus I course. While the textbook has lots of good information and problems, our standards for what constitutes a complete, correct answer are a little different from the answers offered in the textbook. We are providing this document as a resource to help you understand our expectations. This is a supplement to this section of the textbook, not a replacement.

Imagine that you are riding in a car, and take a picture of the speedometer. Looking at the speedometer, you see that it reads 30 mph. What does it *mean* that you were traveling 30 mph at that *instant*? Your velocity is the rate of change of your position with respect to time, but at that instant, your position didn't change at all. So how do we explain what the 30 mph actually means?

This is the sort of question we consider in this section. We're going to work with a different example before coming back to the speedometer.

Example 1

Consider the following question, #3 from §2.4:

3. The temperature, T , in degrees Fahrenheit, of a cold yam placed in a hot oven is given by $T = f(t)$, where t is the time in minutes since the yam was put in the oven. ... What is the practical meaning of the statement $f'(20) = 2$?

Here is an example of the sort of sentence we want to see:

During the twenty-first minute after the cold yam was placed in the hot oven, the temperature increased by about 2 degrees Fahrenheit.

The key features that make this a complete, correct answer include the following:

- The answer is a complete, comprehensible English sentence that could be understood by someone who knows no calculus. (This sentence should not rely on the word “rate”.)
- The given input value is included.
- A specific, small change in input variable from that input value is stated.
- The appropriate corresponding change in output value is stated.
- The appropriate direction of change of output (such as “increase” or “decrease”) is given. Note: take care not to say that an output is increasing by a negative amount. We want your sentence to be written in a way someone might actually say it.
- It is made clear that the change in the output is approximate (“about”, “approximately”, “roughly”, etc.).
- Correct units are used everywhere it is appropriate.
- The sentence indicates that this approximation is only happening near the input value. (You should avoid phrases such as “for all” or “for every” or the word “per” to describe the derivative.)

How did we come to this answer? We'll explore this by starting with some typical answers we often see and identifying why these answers are incorrect. In this document, answers that would receive full credit are given in **bold green** while answers that would not receive full credit are given in *italicized blue*.

Like any problem in context, one useful first step is to consider units. Since 20 is the input for $f'(t)$, it represents the value $t = 20$, that is, 20 minutes after the cold yam was placed in the hot oven. *Leibniz notation* is useful for thinking about the units for the 2, which is a value of $f'(t)$. In Leibniz notation,

$$f'(t) = \frac{dT}{dt}$$

Since the units of T are **degrees Fahrenheit** and the units of t are **minutes**, the units of $f'(t)$ are **degrees Fahrenheit per minute**.

With this analysis, we might come up with the following interpretation:

Twenty minutes after the cold yam was placed in the hot oven, the temperature of the yam is increasing at a rate of 2 degrees Fahrenheit per minute.

But this has the same problem as the speedometer: what does it *mean* for a function to have a particular rate of change at a *single input value*? This section is about thinking of derivatives as rates of change, and we want to see you give an answer that acknowledges this notion of *change*.

So in the case of this example, perhaps we would say:

Twenty minutes after the cold yam was placed in the hot oven, the temperature of the yam is increasing by 2 degrees Fahrenheit each minute.

But this isn't quite accurate. It seems to be implying that the rate of change is a constant 2 degrees Fahrenheit per minute. We have no reason to think that the rate of change will be the same for t -values other than $t = 20$. One way to correct this would be to acknowledge the assumption we're making:

Twenty minutes after the cold yam was placed in the hot oven, the temperature of the yam would be increasing by 2 degrees Fahrenheit each minute if it were changing at a constant rate.

This statement is true, but isn't a particularly practical (or useful) interpretation, because it's unlikely to be the case that the rate of change of the temperature of the yam is constant. Instead, let's try:

During the twenty-first minute after the cold yam was placed in the hot oven, the temperature increased by 2 degrees Fahrenheit.

This is better, since we're only talking about what happens relatively near $t = 20$, but it still isn't entirely accurate. Most likely, the derivative isn't constant even during that single minute, so we need to make it clear that this is only an *approximation* for the actual change. Here's a good interpretation that takes all the ideas above into consideration:

During the twenty-first minute after the cold yam was placed in the hot oven, the temperature increased by about 2 degrees Fahrenheit.

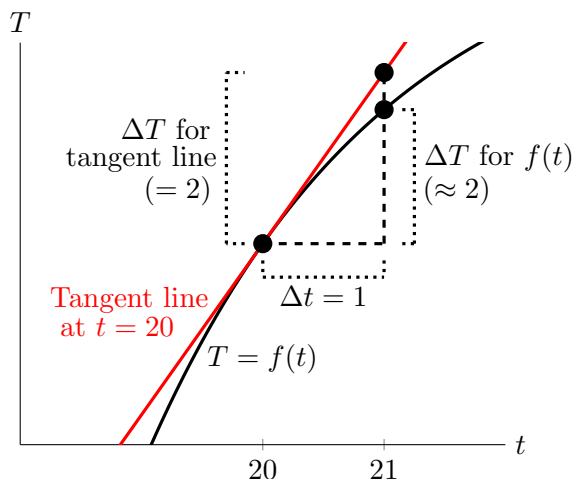
This is just one example of many sentences that would work. Here are some more:

- **After it has been in the oven for 20 minutes, the yam's temperature will increase by approximately $1/3^\circ$ F if it stays in the oven an additional 10 seconds.**
- **If the yam is taken out of the oven 20 minutes after it was put in, it got about 1° F hotter in the last thirty seconds it was in the oven.**
- **The yam was roughly 2° Fahrenheit cooler 19 minutes after it was put into the oven than it was 20 minutes after it was put in the oven.**

If you're not sure how we computed the changed in temperature for all of these sentences, keep reading!

In pictures

Graphically, the idea is as follows:



We are trying to approximate the change in the output T for the function $f(t)$ corresponding (in this case) to a change in the input t of one unit. We will do this by saying the change in T for $f(t)$ is approximately the same as the change in T for the tangent line over the same interval. (That is, ΔT for the tangent line $\approx \Delta T$ for $f(t)$.) This makes sense, because, near $t = 20$, the function looks very similar to its tangent line. Since the tangent line at $t = 20$ has slope $f'(20) = 2$, we know that T will increase by 2 when t increases by 1. Thus, when t increases from 20 to 21, $f(t)$ will increase by about 2.

Earlier in Chapter 2, we used values of a function to approximate its derivative at a point. Now we are using values of the *derivative* to approximate change in values of the *function*. We will do even more with this idea later in Chapter 3.

Example 2

With the function $f(t)$ above, a change of one unit in the input (“during the twenty-first minute”) is probably appropriate. But you should always be careful to think about whether you’re using a reasonably small change in input. For example, let’s go back to our picture of the speedometer reading 30 mph. Recall that velocity is change in position over change in time, or the derivative of the position function. In this case we’ll use $s(t)$ to denote the distance the car has driven, in miles, t hours after starting, and suppose that the picture is taken 10 minutes after starting, i.e. at $t = 10/60 = 1/6$. That means that in this situation we’re asking for an interpretation of the equation $s'(1/6) = 30$. Building an answer similar to the first one from the yam problem, we might end up with:

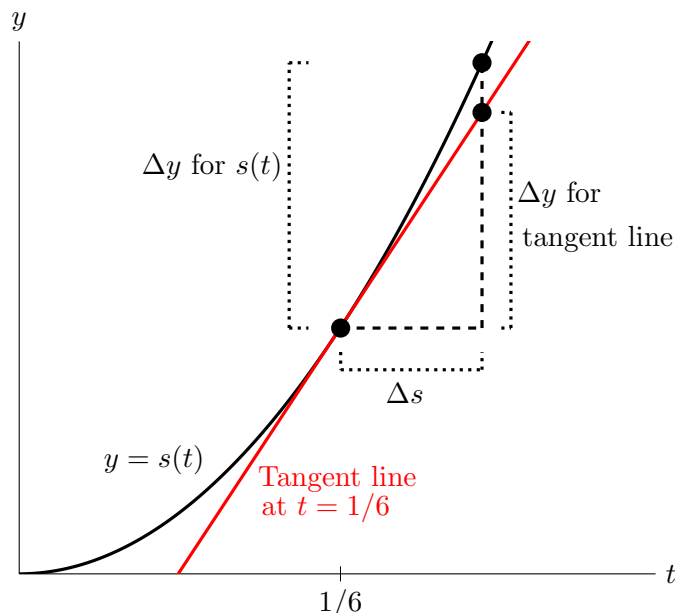
In the hour after the picture is taken, the car drives a distance of about 30 miles.

But is this reasonable? Most likely, the car will not drive 30 mph for that entire hour, meaning our approximation is unlikely to be very accurate. In fact, maybe your trip only lasts 20 minutes, in which case talking about how far the car drives in an hour doesn’t really make sense. So we need to choose a different change in input. You should also be careful not to go too small, though: talking about the change in position during a period of 0.000000000001 hours might give you a very accurate estimate, but it’s not a particularly *practical* estimate. When coming up with an interpretation, try to think about balancing these: use an interval that will both give a reasonably accurate approximation and give information that a person might actually find useful. We can’t give you specific guidelines on this; you’ll need to think through the context carefully and decide for yourself.

In this example, let’s use a change in input of one minute, that is, $\Delta t = 1/60$. With this change in input, the change in output is no longer 30 (after all, the car won’t move 30 miles in one minute!), so

we need to find the corresponding change of output. To figure out how to do this, let's go back to thinking graphically.

Here is a graph of $y = s(t)$ near $t = 1/6$ with the tangent line at $t = 1/6$ drawn in:



Considering Δy for the tangent line and using Leibniz notation again, we have $s'(1/6) = \frac{\Delta y}{\Delta t}$, which we can rewrite as $\Delta y = s'(1/6)\Delta t$.

In this case, we use $s'(1/6) = 30$ and $\Delta s = 1/60$ to get $\Delta y = 30 \cdot 1/60 = 1/2$. Remember, this is the exact change in y for the tangent line, which is only an approximation for the change in y of $s(t)$. Now we can use this to again make a sentence:

In the minute after the picture is taken, the car drives about half a mile.

Note that we say “minute” instead of “1/60 of an hour”; try to use language that someone would naturally use in conversation.

In class, on Team Homework, and potentially on exams, we will be asking you to generate your own complete sentences, and it's important that you're able to do so. Exams in recent years have often included two additional sorts of problems: ones in which you are given the start of a sentence and asked to complete it, and multiple choice problems. We will look at an example of each here.

Example 3

(From Problem 8(d), Exam 1, Fall 2013.) In Ann Arbor, the average property value P , in dollars per square foot, can be modeled as a function of the distance x , in miles, you are away from the city center. This relationship can be written $P = g(x)$.

Write a sentence expressing the meaning of $g'(0.3) = -160$ which could be understood by someone who knows no calculus. The beginning of the sentence is given below.

If I am 0.3 miles from the center of Ann Arbor looking at properties and I travel 0.05 miles toward the city center, ...

Solution: Note that x is distance from the city center, so traveling 0.05 miles toward the city center actually means that x will *decrease*. That is, we have been given $\Delta x = -0.05$. That means that the

corresponding ΔP will be approximately $g'(0.3) \cdot \Delta x = -160 \cdot -0.05 = 8$. This means that as we move toward the city center, the average property value is actually increasing. We can finish the sentence with:

...the average property value will increase by about \$8 per square foot.

Example 4

Our final example will be a multiple-choice problem. We'll also include another important twist that often appears in these problems: asking you to interpret the meaning of the derivative of the *inverse* of a function.

(From Problem 5, Exam 1, Fall 2014.) Oren, a Math 115 student, realizes that the more caffeine he consumes, the faster he completes his online homework assignments. Let $T(c)$ be the number of minutes it will take Oren to complete tonight's assignment if he consumes c milligrams of caffeine. Suppose that T is continuous and differentiable. Which of the statements below is best supported by the equation $(T^{-1})'(20) = -10$? Circle the one best answer.

- i. If Oren has consumed 20 milligrams of caffeine, then consuming an additional milligram of caffeine will save him about 10 minutes on tonight's assignment.
- ii. The amount of caffeine that will result in Oren finishing his homework in 21 minutes is approximately 10 milligrams greater than the amount of caffeine that Oren will need in order to finish his homework in 20 minutes.
- iii. The rate at which Oren is consuming caffeine 20 minutes into his homework assignment is decreasing by 10 milligrams per minute.
- iv. In order to complete tonight's assignment in 19 rather than 20 minutes, Oren needs to consume about 10 milligrams of additional caffeine.
- v. If Oren consumes 20 milligrams of caffeine, then he will finish tonight's assignment approximately 10 minutes faster than if he consumes no caffeine.

Solution: Just as we did with the yam problem, a good place to start with a problem like this is to consider units. Since the input of $T(c)$ is milligrams of caffeine, and the output is minutes, T^{-1} will reverse this: its input is in minutes, and the output is the amount of caffeine, in milligrams, Oren should consume in order to complete his homework in that number of minutes.

Therefore we are looking for an answer involving 20 minutes. This rules out options i. and v.

We can also rule out iii. Even though the 20 refers to minutes here, it doesn't refer to *the number of minutes it takes him to complete his homework*, as it should, based on the meaning of T .

This leaves us with ii. and iv. Now let's look more closely at the units on -10 to get a better sense of what the derivative tells us. Since the output of T^{-1} is milligrams and the input is minutes, the units on $T^{-1}(20)$ are milligrams per minute. The basic idea of what this derivative is telling us, then, is that a change of -10 milligrams corresponds to a change of 1 minute. Adding in more of the context of the problem, this is saying that 10 milligrams less of caffeine means one more minute of time spent on homework. Considering ii. and iv., which better matches this statement?

Sentence ii. is saying that an increase of c from 20 to 21 will correspond to about 10 milligrams *more* of caffeine, which does not match what we found above. So let's look at iv. This sentence says that a change of c from 20 to 19—a decrease by one unit—corresponds to an increase in caffeine of about 10 milligrams. As in Example 3, when we decrease the input and have a negative derivative, this should

correspond to a positive change in output, which is what's given in this sentence. Therefore, iv. is the correct answer.

Another important thing to notice in this example is that, even though the input is time, it doesn't make sense to write a sentence like

As the amount of time it takes Oren to complete tonight's homework assignment increases from 19 to 20 minutes, he will consume about 10 milligrams less of caffeine.

He can't consume less caffeine, and the amount of time it takes him to complete tonight's homework assignment can only have one value. Keep an eye out for contexts like this, where we're not in a scenario where the input actually changes over time. Instead, we're saying that, depending on what input we choose, we end up with one specific output. Also keep an eye out for problems where time is the output instead of the input (such as for $T(c)$), since it's easy to incorrectly fall back on the assumption that if one of the variables represents time, then it should be the input.

Some final words of caution

Students sometimes try to memorize "formula" sentences for which they can simply fill in the blanks. We recommend against this, for a number of reasons.

- Most importantly: the purpose here is to develop an understanding of the meaning of a derivative. A formulaic approach, even when it works, is not going to facilitate the development of this understanding.
- This approach can lead to some very convoluted-sounding answers that do not resemble something someone would actually say.
- When students memorize a formula, they tend to use a value of 1 for their change of input. This won't work when you're given a different change in input, such as in Example 3, or a change of one unit is inappropriate, such as in Example 2.
- Your formula sentence will likely require oversimplification that will lead to problems. For example, you might assume that if the value of the derivative is negative, then the sentence should always involve a decrease in output, but as you can see in both Examples 3 and 4, that depends on how the input changes.
- Another oversimplification problem: your formula sentence might be something like "When [input] changes from [value] to [value]," but as you can see in Example 4, this formulation is not always appropriate for the context.
- In multiple choice problems like Example 4, it may be the case that none of the options will be in exactly the format you have memorized. You need to be able to evaluate whether sentences in a variety of forms are appropriate.

Instead, take the time to make sense of what's going on in the problem, and use that understanding to either write your answer or choose an appropriate multiple-choice answer.