Math 116 — Practice for Exam 2

Generated March 23, 2025

Name: ____

Instructor: ____________________ Section Number: ________

1. This exam has 10 questions. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
2. Do not separate the pages of the exam. If any pages do become separated, write your name on them and point them out to your instructor when you hand in the exam.
3. Please read the instructions for each individual exercise carefully. One of the skills being tested on this exam is your ability to interpret questions, so instructors will not answer questions about exam problems during the exam.
4. Show an appropriate amount of work (including appropriate explanation) for each exercise so that the graders can see not only the answer but also how you obtained it. Include units in your answers where appropriate.
5. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
6. If you use graphs or tables to obtain an answer, be certain to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
7. You must use the methods learned in this course to solve all problems.

Semester	Exam	Problem	Name	Points	Score
Fall 2021	2	9	gardening	20	
Fall 2013	3	9	olive oil	13	
Fall 2023	1	7	Sisyphus	7	
Winter 2016	2	9		12	
Winter 2003	3	9	nautilus	6	
Fall 2019	2	4	spider web	8	
Fall 2008	3	5		12	
Fall 2017	2	7	bouncy ball	12	
Winter 2011	3	4	signal fire	8	
Fall 2011	3	7		12	
Total				110	

Recommended time (based on points): 114 minutes

9. [20 points] Otto would like to landscape his yard, so he contacts the company Granville's Calculate-Yourself Gardening. Granville's company provides each potential customer with a list of possible equations and improper integrals guiding the landscaping of the yard. Granville's company also offers a discount to customers that can correctly solve the integrals. Otto, strapped for cash, desperately wants to solve the equations Granville has sent him.

a. [3 points] Granville has also informed Otto that he can plant a maple tree in the back of the yard that is initially 2 meters tall. Granville estimates that the maple tree will grow at an instantaneous rate of

$$M(t) = \frac{12t}{e^t}$$

meters per year t years after it is planted. Write an integral that gives the height of the tree t years after it is planted. Your answer should not involve the letter M.

b. [7 points] Determine the maximum height that the maple tree will grow to.

9. (continued)

c. [10 points] Granville tells Otto that he can get a truck to come and dispense dirt into the yard for 2 hours. The instantaneous rate that the truck will dispense dirt t hours after the truck arrives is

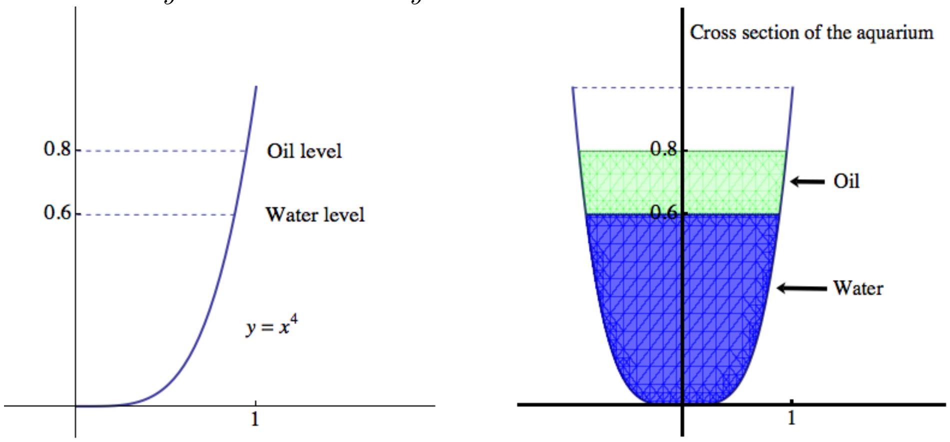
$$D(t) = \frac{(\sin(t))^2}{t^{5/2}\sqrt{2-t}}$$

pounds per minute. Show that $\displaystyle\int_0^2 D(t)dt$ converges. Justify all of your work.

Hint 1: Try splitting this into 2 integrals, one from 0 to 1, and the other from 1 to 2.

Hint 2: You may want to use the fact that $\sin t \leq t$ *for* $t \geq 0$.

9. [13 points] Olive oil has been poured into the Math Department's starfish aquarium! The shape of the aquarium is a solid of revolution, obtained by rotating the graph of $y = x^4$ for $0 \le x \le 1$ around the y-axis. Here x and y are measured in meters.



The aquarium contains water up to a level of $y = 0.6$ meters. There is a layer of oil of thickness 0.2 meters floating on top of the water. The water and olive oil have densities 1000 and 800 kg per m^3, respectively. Use the value of $g = 9.8$ m per s^2 for the acceleration due to gravity.

a. [6 points] Give an expression involving definite integrals that computes the total mass of the water in the aquarium.

b. [7 points] Give an expression involving definite integrals that computes the work necessary to pump all the olive oil to the top of the aquarium.

7. [7 points] Not content with rolling a whole boulder up a hill for all of eternity, Sisyphus instead opts to break up his punishment boulder into smaller pieces of rock and lift them up the hill inside a bucket.

Suppose Sisyphus builds a platform at the top of the hill that is 15 feet above the ground. He lifts the bucket vertically from ground level to the platform. Unfortunately, the bucket has a hole where rocks can fall out.

a. [3 points] Let $W(y)$ be the weight of the bucket with rocks, in pounds, when it is y feet **above the ground**. Write an expression involving one or more integrals for the total work done to lift the bucket up to the platform. Your answer should involve $W(y)$. Do not evaluate your integral(s). Include units.

Answer: ______________________________ **Units:** ____________

b. [4 points] Sisyphus lifts the bucket up at a constant rate of 2 feet per second. The weight of the bucket with rocks decreases at a rate of

$$r(t) = \frac{10}{1 + e^{-t}}$$

pounds per second, where t is measured in seconds since Sisyphus started lifting the bucket. Assume the bucket and the rocks together weigh 100 pounds initially. Find a formula for $W(y)$ involving one or more integrals. Do not evaluate your integral(s).

Answer: ______________________________

9. [12 points]

a. [6 points] Show that the following integral diverges. Give full evidence supporting your answer, showing all your work and indicating any theorems about improper integrals you use.

$$\int_1^\infty \frac{\cos\left(\frac{1}{t}\right)}{\sqrt{t}}\, dt$$

b. [6 points] Find the limit

$$\lim_{x\to\infty} \frac{\int_1^x \frac{\cos\left(\frac{1}{t}\right)}{\sqrt{t}}\, dt}{\sqrt{x}}$$

9. (6 pts) The chambered nautilus builds a spiral sequence of closed chambers. It constructs them from the inside out, with each chamber approximately 20% larger (by volume) than the last. (The large open section at the top is not a "chamber.") The largest chamber is 9 cubic inches. How much volume is enclosed by all the chambers? Assume for simplicity that there are infinitely many chambers. Show your work.

4. [8 points] The amount of time it takes a spider to build a web is t hours. The **cumulative distribution function** for t is given by:

$$J(t) = \begin{cases} 0, & \text{for } t \le \frac{1}{2} \\ \frac{16}{9}\left(-\frac{1}{3}t^3 + \frac{5}{4}t^2 - t + \frac{11}{48}\right), & \text{for } \frac{1}{2} < t < 2 \\ 1, & \text{for } 2 \le t \end{cases}$$

a. [2 points] What appears to be the shortest amount of time it could take the spider to build a web? Include units.

Answer: ______________________

b. [2 points] What is the probability that it will take the spider more than 1 hour to build a web?

Answer: ______________________

c. [4 points] Write an expression for the mean amount of time it takes the spider to build a web. Your answer may involve one or more integrals, but should not involve the letter J.

Answer: ______________________

5. [12 points] For each of the following series, carefully prove its convergence or divergence. You must clearly indicate what test(s) you use in your proof, and must carefully show all work that demonstrates their appropriateness and the calculations associated with the tests.

a. [6 points] $\displaystyle\sum_{n=1}^{\infty} \frac{2^n - 1}{e^n - n}$

b. [6 points] $\displaystyle\sum_{n=2}^{\infty} \frac{n}{n^3 + \cos(n)}$

7. [12 points] A bouncy ball is launched up 20 feet from the floor and then begins bouncing. Each time the ball bounces up from floor, it bounces up again to a height that is 60% the height of the previous bounce. (For example, when it bounces up from the floor after falling 20 ft, the ball will bounce up to a height of $0.6(20) = 12$ feet.)
Consider the following sequences, defined for $n \geq 1$:

- Let h_n be the height, in feet, to which the ball rises when the ball leaves the ground for the nth time. So $h_1 = 20$ and $h_2 = 12$
- Let f_n be the total distance, in feet, that the ball has traveled (both up and down) when it bounces on the ground for the nth time. For example, $f_1 = 40$ and $f_2 = 40 + 24 = 64$.

a. [2 points] Find the values of h_3 and f_3.

Answer: $h_3 =$ ______________ and $f_3 =$ ______________

b. [6 points] Find a closed form expression for h_n and f_n.
("Closed form" here means that your answers should not include sigma notation or ellipses ($\cdots$). Your answers should also **not** involve recursive formulas.)

Answer: $h_n =$ ______________ and $f_n =$ ______________

c. [4 points] Decide whether the given sequence or series converges or diverges.
If it diverges, circle "diverges". If it converges, circle "converges" and write the value to which it converges in the blank.

i. The sequence f_n

Converges to ______________ **Diverges**

ii. The series $\displaystyle\sum_{n=1}^{\infty} h_n$

Converges to ______________ **Diverges**

4. [8 points] You are trapped on an island, and decide to build a signal fire to alert passing ships. You start the fire with 200 pounds of wood. During the course of a day, 40% of the wood pile burns away (so 60% remains). At the end of each day, you add another 200 pounds of wood to the pile. Let W_i denote the weight of the wood pile immediately after adding the i^{th} load of wood (the initial 200-pound pile counts as the first load).

a. [3 points] Find expressions for W_1, W_2 and W_3.

b. [3 points] Find a closed form expression for W_n (*a closed form* expression means that your answer should not contain a large summation).

c. [2 points] Instead of starting with 200 pounds of wood and adding 200 pounds every day, you decide to start with P pounds of wood and add P pounds every day. If you plan to continue the fire indefinitely, determine the largest value of P for which the weight of the wood pile will never exceed 1000 pounds.

7. [12 points] Determine whether the following series converge or diverge (circle your answer). Be sure to mention which tests you used to justify your answer. If you use the comparison test or limit comparison test, write an appropriate comparison function.

a. [3 points] $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{\sqrt{n}}{1+2\sqrt{n}}$

b. [4 points] $\displaystyle\sum_{n=1}^{\infty} ne^{-n^2}$

c. [5 points] $\displaystyle\sum_{n=1}^{\infty}\frac{\cos(n^2)}{n^2}$