Math 116 — Practice for Exam 2

Generated March 23, 2025

Name: SOLUTIONS

Instructor: ______________________ Section Number: __________

1. This exam has 10 questions. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
2. Do not separate the pages of the exam. If any pages do become separated, write your name on them and point them out to your instructor when you hand in the exam.
3. Please read the instructions for each individual exercise carefully. One of the skills being tested on this exam is your ability to interpret questions, so instructors will not answer questions about exam problems during the exam.
4. Show an appropriate amount of work (including appropriate explanation) for each exercise so that the graders can see not only the answer but also how you obtained it. Include units in your answers where appropriate.
5. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
6. If you use graphs or tables to obtain an answer, be certain to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
7. You must use the methods learned in this course to solve all problems.

Semester	Exam	Problem	Name	Points	Score
Fall 2021	2	9	gardening	20	
Fall 2013	3	9	olive oil	13	
Fall 2023	1	7	Sisyphus	7	
Winter 2016	2	9		12	
Winter 2003	3	9	nautilus	6	
Fall 2019	2	4	spider web	8	
Fall 2008	3	5		12	
Fall 2017	2	7	bouncy ball	12	
Winter 2011	3	4	signal fire	8	
Fall 2011	3	7		12	
Total				110	

Recommended time (based on points): 114 minutes

9. [20 points] Otto would like to landscape his yard, so he contacts the company Granville's Calculate-Yourself Gardening. Granville's company provides each potential customer with a list of possible equations and improper integrals guiding the landscaping of the yard. Granville's company also offers a discount to customers that can correctly solve the integrals. Otto, strapped for cash, desperately wants to solve the equations Granville has sent him.

a. [3 points] Granville has also informed Otto that he can plant a maple tree in the back of the yard that is initially 2 meters tall. Granville estimates that the maple tree will grow at an instantaneous rate of

$$M(t) = \frac{12t}{e^t}$$

meters per year t years after it is planted. Write an integral that gives the height of the tree t years after it is planted. Your answer should not involve the letter M.

Solution:

$$2 + \int_0^t \frac{12s}{e^s} ds.$$

b. [7 points] Determine the maximum height that the maple tree will grow to.

Solution: Let us first calculate the integral in (a) using integration by parts ($u = 12s, dv = e^{-s}ds$):

$$\begin{aligned}\int_0^t \frac{12s}{e^s} ds &= -12se^{-s}\big|_0^t + 12\int_0^t e^{-s} ds \\ &= -12te^{-t} + 12\Big[-e^{-s}\big|_0^t\Big] \\ &= -12te^{-t} - 12e^{-t} + 12\end{aligned}$$

Since the tree is always growing, the maximum height is

$$\begin{aligned}2 + \int_0^\infty \frac{12t}{e^t} dt &= 2 + \lim_{t\to\infty} \int_0^t \frac{12s}{e^s} ds \\ &= 2 + \lim_{t\to\infty} 12 - 12e^{-t} - 12te^{-t} \\ &= 14 - \lim_{t\to\infty} \frac{12t}{e^t} \\ &\overset{\text{L'H}\frac{\infty}{\infty}}{=} 14 - \lim_{t\to\infty} \frac{12}{e^t} \\ &= 14.\end{aligned}$$

So, the maximum height of the tree is 12 meters.

9. (continued)

c. [10 points] Granville tells Otto that he can get a truck to come and dispense dirt into the yard for 2 hours. The instantaneous rate that the truck will dispense dirt t hours after the truck arrives is

$$D(t) = \frac{(\sin(t))^2}{t^{5/2}\sqrt{2-t}}$$

pounds per minute. Show that $\int_0^2 D(t)dt$ converges. Justify all of your work.

Hint 1: Try splitting this into 2 integrals, one from 0 to 1, and the other from 1 to 2.

Hint 2: You may want to use the fact that $\sin t \leq t$ *for* $t \geq 0$.

Solution: Following the hint,

$$\int_0^2 D(t)dt = \int_0^1 D(t)dt + \int_1^2 D(t)dt.$$

Starting with the first integral, note that on $[0,1]$,

$$D(t) \leq \frac{t^2}{t^{5/2}} \leq t^{-1/2}$$

By the p-test ($p = \frac{1}{2}$), $\int_0^1 \frac{1}{\sqrt{t}}dt$ converges, so by the comparison test, $\int_0^1 D(t)dt$ converges.

Next, we perform a change of variables on the second integral($w = 2-t$):

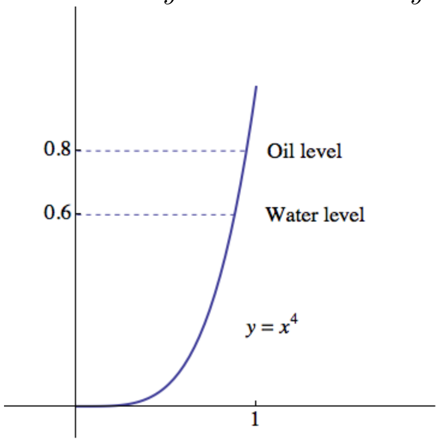
$$\begin{aligned}
\int_1^2 D(t)dt &= \lim_{b\to 2}\int_1^b D(t)dt \\
&= \lim_{b\to 2} -\int_1^{2-b} \frac{(\sin(2-w))^2}{(2-w)^{5/2}\sqrt{w}}dw \\
&= \int_0^1 \frac{(\sin(2-w))^2}{(2-w)^{5/2}\sqrt{w}}dw
\end{aligned}$$

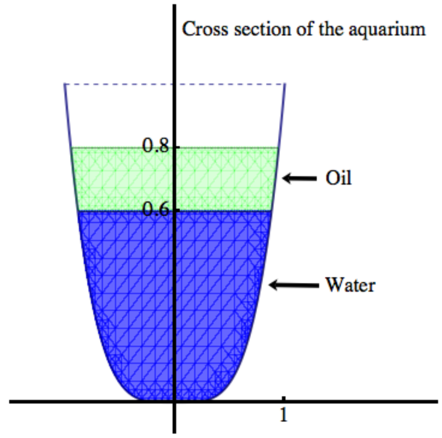
On the interval $[0,1]$,

$$\frac{(\sin(2-w))^2}{(2-w)^{5/2}\sqrt{w}} \leq \frac{1}{\sqrt{w}}$$

By the p-test ($p = \frac{1}{2}$), $\int_0^1 \frac{1}{\sqrt{w}}dw$ converges. So, by the comparison test, $\int_1^2 D(t)dt$ converges. As both parts converge, we have verified that the integral $\int_0^2 D(t)dt$ converges.

9. [13 points] Olive oil have been poured into the Math Department's starfish aquarium! The shape of the aquarium is a solid of revolution, obtained by rotating the graph of $y = x^4$ for $0 \leq x \leq 1$ around the y-axis. Here x and y are measured in meters.





The aquarium contains water up to a level of $y = 0.6$ meters. There is a layer of oil of thickness 0.2 meters floating on top of the water. The water and olive oil have densities 1000 and 800 kg per m^3, respectively. Use the value of $g = 9.8$ m per s^2 for the acceleration due to gravity.

a. [6 points] Give an expression involving definite integrals that computes the total mass of the water in the aquarium.

Solution: $\text{Mass}_{water} = \int_0^{0.6} \pi(\sqrt[4]{y})^2(1000)dy = \int_0^{0.6} \pi\sqrt{y}(1000)dy$

b. [7 points] Give an expression involving definite integrals that computes the work necessary to pump all the olive oil to the top of the aquarium.

Solution: $\text{Work}_{oil} = \int_{0.6}^{0.8} \pi(\sqrt[4]{y})^2(800)(9.8)(1-y)dy = \int_{0.6}^{0.8} \pi\sqrt{y}(800)(9.8)(1-y)dy$

7. [7 points] Not content with rolling a whole boulder up a hill for all of eternity, Sisyphus instead opts to break up his punishment boulder into smaller pieces of rock and lift them up the hill inside a bucket.

Suppose Sisyphus builds a platform at the top of the hill that is 15 feet above the ground. He lifts the bucket vertically from ground level to the platform. Unfortunately, the bucket has a hole where rocks can fall out.

a. [3 points] Let $W(y)$ be the weight of the bucket with rocks, in pounds, when it is y feet **above the ground**. Write an expression involving one or more integrals for the total work done to lift the bucket up to the platform. Your answer should involve $W(y)$. Do not evaluate your integral(s). Include units.

Answer: $\int_0^{15} W(y)\, dy$ **Units:** ftlbs

b. [4 points] Sisyphus lifts the bucket up at a constant rate of 2 feet per second. The weight of the bucket with rocks decreases at a rate of

$$r(t) = \frac{10}{1 + e^{-t}}$$

pounds per second, where t is measured in seconds since Sisyphus started lifting the bucket. Assume the bucket and the rocks together weigh 100 pounds initially. Find a formula for $W(y)$ involving one or more integrals. Do not evaluate your integral(s).

Solution: As the bucket is lifted at a constant rate, the height of the bucket after t seconds is $y = 2t$, and so $t = \frac{y}{2}$. The total decrease in weight in the first $\frac{y}{2}$ seconds is given by integrating the rate of change from 0 to $\frac{y}{2}$. Therefore, remembering the initial weight is 100 pounds, we see that the weight after lifting the bucket y feet is given by

$$W(y) = 100 - \int_0^{y/2} \frac{10}{1 + e^{-t}}\, dt$$

Answer: $W(y) = 100 - \int_0^{y/2} \frac{10}{1 + e^{-t}}\, dt$

9. [12 points]

a. [6 points] Show that the following integral diverges. Give full evidence supporting your answer, showing all your work and indicating any theorems about improper integrals you use.

$$\int_1^\infty \frac{\cos(\frac{1}{t})}{\sqrt{t}}\, dt$$

Solution: $t \geq 1 \Rightarrow \frac{1}{t} \leq 1 \Rightarrow \cos(\frac{1}{t}) \geq \cos(1)$ because the function $F(x) = \cos x$ is decreasing in the interval $[0, 1]$. Therefore,

$$\frac{\cos(\frac{1}{t})}{\sqrt{t}} \geq \frac{\cos(1)}{\sqrt{t}}$$

The improper integral

$$\int_1^\infty \frac{\cos(1)}{\sqrt{t}}\, dt = \cos(1) \int_1^\infty \frac{1}{\sqrt{t}}\, dt$$

diverges by the p-test since $p = \frac{1}{2} \leq 1$. So the integral

$$\int_1^\infty \frac{\cos(\frac{1}{t})}{\sqrt{t}}\, dt$$

diverges by the comparison test (notice that $\cos(1) > 0$).

b. [6 points] Find the limit

$$\lim_{x\to\infty} \frac{\int_1^x \frac{\cos(\frac{1}{t})}{\sqrt{t}}\, dt}{\sqrt{x}}$$

Solution: Notice that by (a), this is $\frac{\infty}{\infty}$. We use L'Hopital's rule along with the 2nd Fundamental Theorem in the numerator:

$$\lim_{x\to\infty} \frac{\int_1^x \frac{\cos(\frac{1}{t})}{\sqrt{t}}\, dt}{\sqrt{x}} = \lim_{x\to\infty} \frac{\frac{\cos(\frac{1}{x})}{\sqrt{x}}}{\frac{1}{2\sqrt{x}}} = \lim_{x\to\infty} 2\cos\left(\frac{1}{x}\right) = 2\cos(0) = 2$$

9. (6 pts) The chambered nautilus builds a spiral sequence of closed chambers. It constructs them from the inside out, with each chamber approximately 20% larger (by volume) than the last. (The large open section at the top is not a "chamber.") The largest chamber is 9 cubic inches. How much volume is enclosed by all the chambers? Assume for simplicity that there are infinitely many chambers. Show your work.

Because the chambers grow by a constant factor each time, they form a geometric series. If each is 20% larger than the previous, then the ratio between them is 1.2. But this is the ratio of the larger divided by the smaller, and we want the opposite, so we get $r = 1/1.2 = 5/6$. This is the ratio by which you have to multiply each volume to get the next smaller volume. The total volume, then, is:

$$9 + 9(\frac{5}{6}) + 9(\frac{5}{6})^2 + 9(\frac{5}{6})^3 \ldots.$$

This geometric series sums to $\frac{9}{1-\frac{5}{6}} = 54$. So the total enclosed volume is 54 cubic inches.

By the way, the numbers given in this problem are not simply made up, but are deduced from the size and shape of a large adult chambered nautilus. The number 54 is the approximate volume of a cylinder with height 2 inches and radius 3 inches (a rough approximation to the organism's size and shape).

Where does $5/6$ come from? Notice that one "band" of the chambers takes about 17 chambers, and (by directly measuring the picture), shrinks the organism by a factor of 3, *in length*. Scaling down by a factor of 3 in length is the same as scaling by a factor of 27 in volume, which should leave $54/27 = 2$ cubic inches. Therefore the first 17 chambers take 52 cubic inches. So we have the equations:

$$\frac{a}{1-r} = 54 \text{ and } \frac{a(1-r^{17})}{1-r} = 52.$$

Solving simultaneously gives $a = 9.5 \approx 9$, $r = .82 \approx 5/6$. This is how the problem was written.

4. [8 points] The amount of time it takes a spider to build a web is t hours. The **cumulative distribution function** for t is given by:

$$J(t) = \begin{cases} 0, & \text{for } t \leq \frac{1}{2} \\ \frac{16}{9}\left(-\frac{1}{3}t^3 + \frac{5}{4}t^2 - t + \frac{11}{48}\right), & \text{for } \frac{1}{2} < t < 2 \\ 1, & \text{for } 2 \leq t \end{cases}$$

a. [2 points] What appears to be the shortest amount of time it could take the spider to build a web? Include units.

Answer: $\frac{1}{2}$ **hour**

b. [2 points] What is the probability that it will take the spider more than 1 hour to build a web?

Answer: $1-\frac{7}{27} \approx 0.74074$

c. [4 points] Write an expression for the mean amount of time it takes the spider to build a web. Your answer may involve one or more integrals, but should not involve the letter J.

Answer: $\int_{1/2}^{2} \frac{16t}{9}\left(-t^2 + \frac{5}{2}t - 1\right) dt$

5. [12 points] For each of the following series, carefully prove its convergence or divergence. You must clearly indicate what test(s) you use in your proof, and must carefully show all work that demonstrates their appropriateness and the calculations associated with the tests.

a. [6 points] $\displaystyle\sum_{n=1}^{\infty} \frac{2^n - 1}{e^n - n}$

Solution: There are a couple of possible methods we could use to show that this series converges. Using the ratio test, we look at $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$. This is

$$\lim_{n\to\infty} \left|\frac{2^{n+1}-1}{e^{n+1}-(n+1)} \cdot \frac{e^n - n}{2^n - 1}\right| = \lim_{n\to\infty} \left|\frac{e^n 2^n (2 - \frac{1}{2^n})(1 - \frac{n}{e^n})}{e^n 2^n (e - \frac{n+1}{e^n})(1 - \frac{1}{2^n})}\right| = \frac{2}{e}.$$

The limit is less than one, so by the ratio test we know that this series converges. Alternately, we could use the limit comparison test and compare with the geometric series $\sum \left(\frac{2}{e}\right)^n$; $\frac{2}{e} < 1$, so this is a convergent geometric series. Then we look at $\lim_{n\to\infty} \frac{b_n}{a_n}$, which is

$$\lim_{n\to\infty} \frac{2^n - 1}{e^n - n} \cdot \frac{e^n}{2^n} = \lim_{n\to\infty} \frac{2^n e^n (1 - \frac{1}{e^n})}{2^n e^n (1 - \frac{n}{e^n})} = 1.$$

This is finite, so the convergent properties of the two are the same, and thus our series converges.

b. [6 points] $\displaystyle\sum_{n=2}^{\infty} \frac{n}{n^3 + \cos(n)}$

Solution: Note that $a_n = \frac{n}{n^3+\cos(n)} \le \frac{n}{n^3-1} < \frac{n}{n^3 - \frac{1}{4}n^3} = \frac{4n}{3n^3} = \frac{4}{3n^2}$ for $n \ge 2$. Further $a_n > 0$. Thus, by comparison with the series $\frac{4}{3}\sum \frac{1}{n^2}$, which we know converges, we must have that this series converges.

Alternately, we could also use limit comparison with $\sum \frac{1}{n^2}$, which we know converges. The terms of the given and comparison series are positive, so we can use limit comparison. Then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^3}{n^3 + \cos(n)} = \lim_{n\to\infty} \frac{1}{1 + \frac{\cos(n)}{n^3}} = 1.$$

Thus, by the limit comparison test we know that $\sum \frac{n}{n^3+\cos(n)}$ must converge.

7. [12 points] A bouncy ball is launched up 20 feet from the floor and then begins bouncing. Each time the ball bounces up from floor, it bounces up again to a height that is 60% the height of the previous bounce. (For example, when it bounces up from the floor after falling 20 ft, the ball will bounce up to a height of $0.6(20) = 12$ feet.)
Consider the following sequences, defined for $n \geq 1$:

- Let h_n be the height, in feet, to which the ball rises when the ball leaves the ground for the nth time. So $h_1 = 20$ and $h_2 = 12$
- Let f_n be the total distance, in feet, that the ball has traveled (both up and down) when it bounces on the ground for the nth time. For example, $f_1 = 40$ and $f_2 = 40 + 24 = 64$.

a. [2 points] Find the values of h_3 and f_3.

Solution: $h_3 = 0.6(12) = 7.2$ and $f_3 = 64 + 14.4 = 78.4$.

Answer: $h_3 =$ 7.2 and $f_3 =$ 78.4

b. [6 points] Find a closed form expression for h_n and f_n.
("Closed form" here means that your answers should not include sigma notation or ellipses $(\cdots)$. Your answers should also **not** involve recursive formulas.)

Solution: $h_n = 0.6h_{n-1}$ is a recursive relationship that holds between the terms of the sequence h_n for $n > 1$, and this recursive formula means that h_n is a geometric sequence. The (constant) ratio of successive terms is equal to 0.6 and first term is $h_1 = 20$. So we see that $h_n = 20(0.6)^{n-1}$.

Note that the term f_n is twice the sum of the first n terms of the h_n sequence. (Twice because the bouncy ball travels both up and down.) We use the formula for a partial sum of a geometric series (i.e. a finite geometric series) to find

$$\begin{aligned} f_n &= 2(h_1 + h_2 + \ldots + h_n) = 2(20 + \ldots + 20(0.6)^{n-1}) \\ &= \frac{2(20)(1-(0.6)^n)}{1-0.6} = \frac{40(1-(0.6)^n)}{0.4} = 100(1-(0.6)^n). \end{aligned}$$

Answer: $h_n =$ $20 \cdot (0.6)^{n-1}$ and $f_n =$ $\frac{40(1-(0.6)^n)}{0.4} = 100(1-(0.6)^n)$

c. [4 points] Decide whether the given sequence or series converges or diverges.
If it diverges, circle "diverges". If it converges, circle "converges" and write the value to which it converges in the blank.

i. The sequence f_n

Converges to 100 **Diverges**

Solution: The limit of the sequence f_n is

$$\lim_{n\to\infty} f_n = \lim_{n\to\infty} \frac{40(1-(0.6)^n)}{0.4} = \frac{40}{0.4} = 100.$$

Since this limit exists, the sequence f_n converges, and this computation shows that it converges to 100.

Alternatively, as we saw in part **b**, the sequence f_n is the sequence of partial sums of the geometric series $\sum_{k=1}^{\infty} 2h_k = \sum_{k=1}^{\infty} 40(0.6)^{k-1}$. Since $r = 0.6$ and $|0.6| < 1$, we know that this geometric series converges to $\frac{40}{1-0.6} = 100$. By definition of series convergence, this sum is the limit of the sequence of partial sums f_n, i.e. $\lim_{n\to\infty} f_n = 100$.

ii. The series $\sum_{n=1}^{\infty} h_n$

Converges to 50 **Diverges**

Solution: Next, we consider the series $\sum_{n=1}^{\infty} h_n$, which we know is geometric from part **b**. Since the common ratio between successive terms is 0.6, the series converges, and the formula for the sum of a convergent geometric series gives us

$$\sum_{n=1}^{\infty} h_n = \sum_{n=1}^{\infty} 20 \cdot (0.6)^{n-1} = \frac{20}{1-0.6} = 50,$$

Alternatively, since the sequence f_n is the sequence of partial sums of the series $\sum_{k=1}^{\infty} 2h_k$, we have $\sum_{n=1}^{\infty} h_n = \frac{1}{2} \lim_{n\to\infty} f_n = \frac{100}{2} = 50.$

4. [8 points] You are trapped on an island, and decide to build a signal fire to alert passing ships. You start the fire with 200 pounds of wood. During the course of a day, 40% of the wood pile burns away (so 60% remains). At the end of each day, you add another 200 pounds of wood to the pile. Let W_i denote the weight of the wood pile immediately after adding the i^{th} load of wood (the initial 200-pound pile counts as the first load).

a. [3 points] Find expressions for W_1, W_2 and W_3.

Solution:

$$W_1 = 200$$
$$W_2 = 200 + 200(0.6)$$
$$W_3 = 200 + 200(0.6) + 200(0.6)^2$$

b. [3 points] Find a closed form expression for W_n (*a closed form* expression means that your answer should not contain a large summation).

Solution:

$$W_n = \frac{200(1 - 0.6^n)}{1 - 0.6}$$

c. [2 points] Instead of starting with 200 pounds of wood and adding 200 pounds every day, you decide to start with P pounds of wood and add P pounds every day. If you plan to continue the fire indefinitely, determine the largest value of P for which the weight of the wood pile will never exceed 1000 pounds.

Solution:

$$\frac{P}{1 - 0.6} = 1000$$
$$P = 400$$

7. [12 points] Determine whether the following series converge or diverge (circle your answer). Be sure to mention which tests you used to justify your answer. If you use the comparison test or limit comparison test, write an appropriate comparison function.

a. [3 points] $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{\sqrt{n}}{1+2\sqrt{n}}$

Solution: Since

$$\lim_{n\to\infty}\frac{\sqrt{n}}{1+2\sqrt{n}}=\lim_{n\to\infty}\frac{\sqrt{n}}{2\sqrt{n}}=\frac{1}{2},$$

then the sequence $a_n=(-1)^n\frac{\sqrt{n}}{1+2\sqrt{n}}$ does not converge to zero (it oscillates closer to $\frac{1}{2}$ and $-\frac{1}{2}$. Since the terms a_n does not converge to 0, then the series $\sum_{n=1}^{\infty}a_n$ diverges.

b. [4 points] $\displaystyle\sum_{n=1}^{\infty}ne^{-n^2}$

Solution: Let $f(x)=xe^{-x^2}$. The function $f(x)>0$ and $f'(x)=e^{-x^2}(1-2x^2)<0$ for $x\geq 1$. Hence by the Integral test

$$\sum_{n=1}^{\infty}ne^{-n^2}\text{ behaves as }\int_1^{\infty}xe^{-x^2}dx=\lim_{b\to\infty}\int_1^b xe^{-x^2}dx=\lim_{b\to\infty}-\frac{1}{2}e^{-x^2}|_1^b=\frac{1}{2e}$$

the series converges.

c. [5 points] $\displaystyle\sum_{n=1}^{\infty}\frac{\cos(n^2)}{n^2}$

Solution: This series has positive and negative terms. Since

$$\left|\frac{\cos(n^2)}{n^2}\right|\leq\frac{1}{n^2},$$

then the series of the absolute values satisfies

$$\sum_{n=1}^{\infty}\left|\frac{\cos(n^2)}{n^2}\right|\leq\sum_{n=1}^{\infty}\frac{1}{n^2}.$$

The series on the right converges by the p series test with $p=2$, hence the series of absolute values converges. Since the series of absolute values converges, then $\sum_{n=1}^{\infty}\frac{\cos(n^2)}{n^2}$ converges.