

MATH 116 — PRACTICE FOR EXAM 3

Generated April 19, 2026

UMID: _____ INITIALS: _____

INSTRUCTOR: _____ SECTION NUMBER: _____

1. This exam has 15 questions. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
2. Please read the instructions for each individual exercise carefully. One of the skills being tested on this exam is your ability to interpret questions, so instructors will not answer questions about exam problems during the exam.
3. Show an appropriate amount of work (including appropriate explanation) for each exercise so that the graders can see not only the answer but also how you obtained it. Include units in your answers where appropriate.
4. You are allowed notes written on two sides of a 3" × 5" note card. You are NOT allowed other resources, including, but not limited to, notes, calculators or other electronic devices.
5. If you use graphs or tables to obtain an answer, be certain to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
6. Problems may ask for answers in exact form. Recall that $x = \sqrt{2}$ is a solution in exact form to the equation $x^2 = 2$, but $x = 1.41421356237$ is not.
7. You must use the methods learned in this course to solve all problems.

Semester	Exam	Problem	Name	Points	Score
Winter 2025	3	6		10	
Fall 2022	2	1		11	
Fall 2016	3	7	Legendre	7	
Fall 2023	3	2		8	
Winter 2011	3	9	Fourier	9	
Winter 2004	2	6	Okefenokee	12	
Fall 2025	3	4	climbing wall	13	
Winter 2025	3	9	park	12	
Winter 2024	1	2		15	
Fall 2023	3	8	vuvuzela	9	
Winter 2019	2	9	Infinity Tower	9	
Winter 2023	2	7	treasure	16	
Fall 2016	1	9	purple key	14	
Fall 2013	1	4	shawarma kafta	17	
Winter 2019	3	5		9	
Total				171	

Recommended time (based on points): 177 minutes

6. [10 points] A power series centered at $x = 3$ is given by

$$\sum_{n=1}^{\infty} \frac{2n+1}{5^n(n^2+1)}(x-3)^n.$$

The radius of convergence of this power series is 5 (do **not** show this). Find the **interval** of convergence of this power series. Show all your work, including full justification for series behavior.

Interval of convergence: _____

1. [11 points]

a. [7 points] Determine the **radius** of convergence of the following power series:

$$\sum_{n=1}^{\infty} \frac{9^n (x-2)^{2n}}{n^2}$$

Be sure to show all of your work. Write your final answer in the space provided below.

Answer: _____.

b. [4 points] Suppose that the power series

$$\sum_{n=1}^{\infty} a_n (x-5)^n$$

converges when $x = 10$ and diverges when $x = -1$. At which of the following x -values must the series converge? Circle your answers. You do not need to show any work for this problem.

-5

0

2

5

11

12

7. [7 points] The *Legendre equation* is a differential equation that arises in the quantum mechanical study of the hydrogen atom. In one of its forms, the Legendre equation is

$$(1 - x^2)y'' - 2xy' + 12y = 0.$$

For this problem, let y be a solution to the Legendre equation satisfying $y(\frac{1}{2}) = 2$ and $y'(\frac{1}{2}) = 3$. Assume that the Taylor series for $y(x)$ about $x = \frac{1}{2}$ converges to $y(x)$ for all $-\frac{1}{2} < x < \frac{3}{2}$.

- a. [4 points] In the blank below, write down $P_2(x)$, the degree 2 Taylor polynomial of $y(x)$ near $x = \frac{1}{2}$. Your answer should not contain the function $y(x)$ or any of its derivatives.

$$P_2(x) = \underline{\hspace{10cm}}$$

- b. [3 points] Compute the limit

$$\lim_{x \rightarrow 1/2} \frac{y(x) - \frac{1}{2} - 3x}{(x - \frac{1}{2})^2}.$$

2. [8 points] Consider the function $G(x) = x^3 \cos(2x)$.
- a. [4 points] Give the first four nonzero terms of the Taylor series of $G(x)$ centered about $x = 0$.

Answer: _____

- b. [4 points] Find $G^{(2023)}(0)$. You do not need to simplify.

Answer: _____

9. [9 points]

a. [2 points] Find the Taylor series about $x = 0$ of $\sin(x^2)$. Your answer should include a formula for the general term in the series.

b. [2 points] Let m be a positive integer, find the Taylor series about $x = 0$ of $\cos(m\pi x)$. Your answer should include a formula for the general term in the series.

c. [5 points] Use the second degree Taylor polynomials of $\sin(x^2)$ and $\cos(m\pi x)$ to approximate the value of b_m , where

$$b_m = \int_{-1}^1 \sin(x^2) \cos(m\pi x) dx.$$

(The number b_m is called a *Fourier coefficient of the function* $\sin x^2$. These numbers play a key role in *Fourier analysis*, a subject with widespread applications in engineering and the sciences.)

6. (12 points) A team of biologists is interested in the ability of certain birds to migrate great distances with little rest. The biologists are monitoring a flock of birds known to migrate after spending the winter in the warm climes of the Okefenokee swamp. The location of the flock, in $x(t)$ hundreds of miles north, and $y(t)$ hundreds of miles east of the base camp of the biologists, t days after their departure from the Okefenokee swamp is given by

$$x(t) = 3t + \frac{1}{2},$$
$$y(t) = t^{\frac{3}{2}} + \frac{1}{5}.$$

(a) Where is the Okefenokee swamp in relation to the base camp of the biologists?

(b) Is there ever a time when the flock of birds is travelling due North-East? If so, when? If not, explain why not.

(c) Is the flock of birds constantly moving throughout the first three days of their journey? Why or why not?

(d) How far does the flock of birds travel in the first three days of their journey?

4. [13 points] A vertical climbing wall is described in coordinates, where (x, y) is the position x meters to the right of a central podium and y meters above the floor. Suppose two climbers, Sara and Tina, begin climbing the wall at the same time. Their positions t minutes after they start climbing are given by:

$$\text{Sara: } \begin{cases} x(t) = -(t-1)^2 \\ y(t) = 3t, \end{cases} \quad \text{Tina: } \begin{cases} x(t) = 1 + \cos(\pi t) \\ y(t) = t^2 + 2. \end{cases}$$

- a. [4 points] Tina and Sara bumped into each other one time during the process of climbing. At what time did this happen? Justify your answer.

Answer: $t =$ _____

- b. [2 points] After Sara and Tina bumped into each other, they climbed for some further distance and eventually crossed the finish line, which is a horizontal line, at the same time. When did they cross the finish line, and what is the equation of the finish line?

Answer: They crossed the finish line at $t =$ _____

Answer: The equation of the finish line is $y =$ _____

- c. [4 points] Write an expression involving one or more integrals that gives the total distance traveled by Tina from the time she starting climbing until she crossed the finish line. Do not evaluate your integral(s). Show your work.

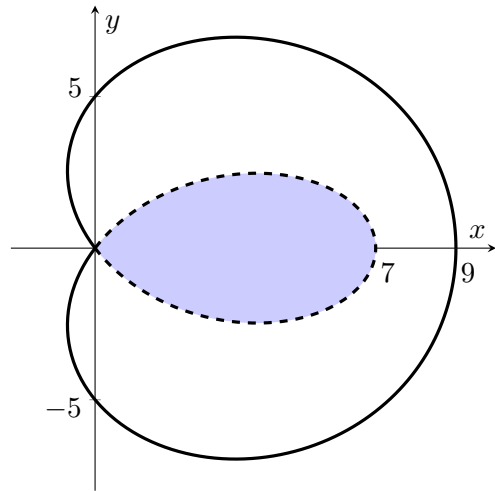
Answer: _____

- d. [3 points] What is Tina's speed at $t = \frac{1}{2}$? **Include units.** You do not need to simplify your answer but you should show your work.

Answer: _____ **Units:** _____

9. [12 points]

Elena, a talented landscape architect, envisions a park whose shape is defined by the polar curve $r = 5 + 8 \cos(\theta) - 4 \cos^2(\theta)$, as illustrated to the right. In her design, the inner loop of the curve serves as an ideal location for a lake, represented by the shaded region. The solid outer curve in the diagram represents the walking trail that winds around the park.



- a. [4 points] Using the factorization $5 + 8 \cos(\theta) - 4 \cos^2(\theta) = (1 + 2 \cos(\theta))(5 - 2 \cos(\theta))$, find the values of θ in the interval $[0, 2\pi)$ for which the curve passes through the origin.

Answer: $\theta =$ _____

- b. [4 points] To determine the amount of water required to fill the lake, Elena wants to calculate the area of the surface of the lake. Write an expression involving one or more integrals that represents the area of the shaded region. Do not evaluate the integral(s).

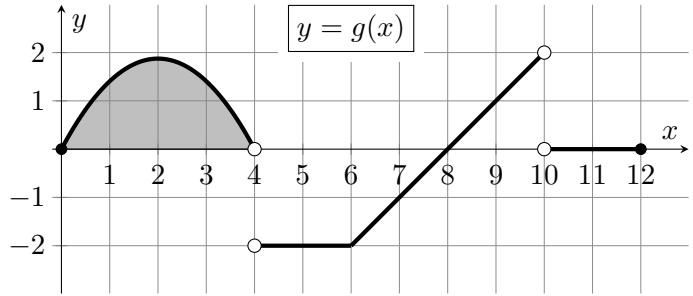
Answer: _____

- c. [4 points] Recall that the solid outer curve in the diagram represents the walking trail. Write an expression involving one or more integrals that represents the total length of the walking trail. Do not evaluate the integral(s).

Answer: _____

2. [15 points] A function $g(x)$ is graphed below and has the following properties:

- $g(x)$ is piecewise linear for $x > 4$.
- The shaded region has area 5.



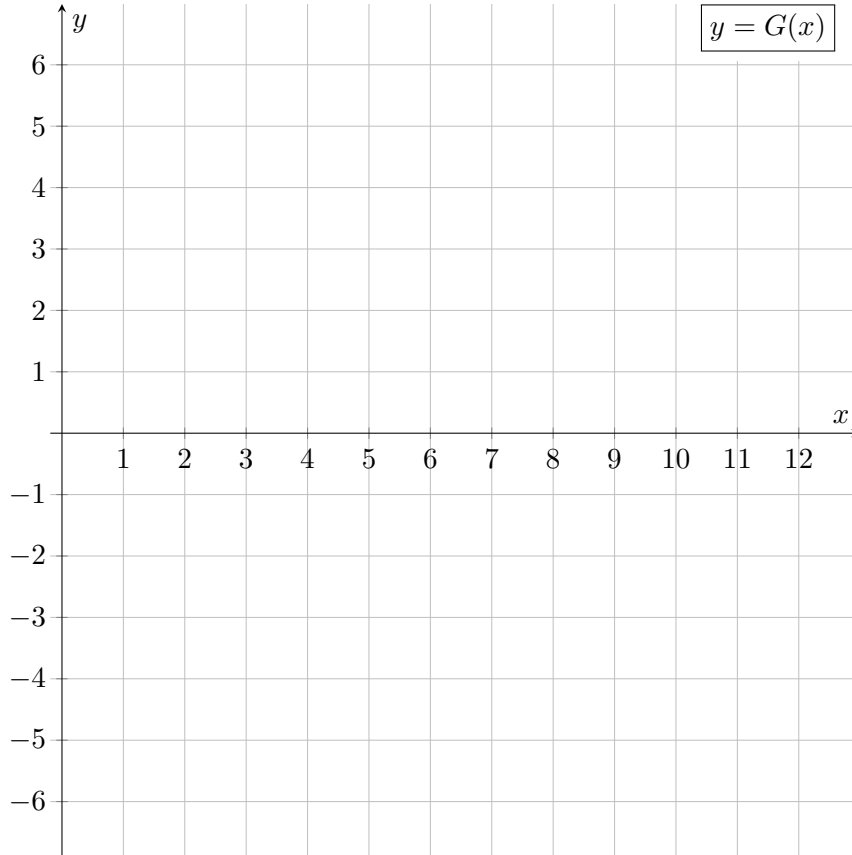
Let $G(x)$ be the continuous antiderivative of $g(x)$ satisfying $G(6) = -1$.

a. [5 points] Use the graph of $g(x)$ to complete the table below with the **exact** values of $G(x)$.

x	0	4	6	8	10	12
$G(x)$			-1			

b. [10 points] Sketch a graph of $G(x)$ on the interval $[0, 12]$ using the axes provided below. Be sure to pay attention to:

- where $G(x)$ is and is not differentiable;
- where $G(x)$ is increasing, decreasing, or constant;
- where $G(x)$ is concave up, concave down, or linear;
- the slope of $G(x)$ at $x = 2$;
- the values of $G(x)$ you found in the table in part a.



8. [9 points] Gabriella is developing a new kind of vuvuzela. In order to come up with a new method, she first considers the old way she made her instruments.
- a. [4 points] Gabriella initially made her vuvuzelas by considering a positive function $f(x)$, and forming a region \mathcal{R} between $y = f(x)$ and the x -axis on the interval $[2, \infty)$. She rotated \mathcal{R} about the x -axis to form the shape of the vuvuzela. Write an integral which gives the volume of the vuvuzela. Your answer will involve the function $f(x)$.

Answer: _____

- b. [5 points] For her new batch of vuvuzelas, Gabriella considers an entirely different shape. The volume of the new design of vuvuzela is given by

$$\int_2^{\infty} \frac{x}{(x^2 + 5)^2} dx.$$

Compute the value of this integral if it converges. If it does not converge, use a **direct computation** of the integral to show its divergence. Be sure to show your full computation, and be sure to use **proper notation**.

Circle one: **Diverges** **Converges to** _____

9. [9 points] The blueprint for the Infinity Tower has been finalized, and the design of the Tower of Hanoi is accepted. Specifically:

- the tower will have infinitely many floors
- each floor has the shape of a solid cylinder of height of 3 meters
- the n th floor has radius $\frac{1}{2n^2}$ meters
- the ground floor corresponds to $n = 1$
- the tower has constant density δ kg/m³
- when construction begins, all materials are on the ground and have to be lifted to build each floor.

In this problem, you may assume the acceleration due to gravity is $g = 9.8$ m/s².

a. [7 points] Let W_n be the work, in Joules, it takes to lift the materials to build the n th floor and put that floor in place in the tower. Write an expression involving one or more integrals for each of the following.

i. $W_1 =$ _____

ii. $W_2 =$ _____

iii. $W_n =$ _____

b. [2 points] Write an expression involving one or more integrals and/or series that gives the total work it would take to build the entire tower. Your answer should not include the letter W .

Answer: _____

7. [16 points] A treasure hunter has spotted a large exotic rock at the bottom of a deep pit. The vertical distance from the top of the pit to the top of the rock is 15 meters. To retrieve the rock, the treasure hunter attaches a 15 meter rope to the top of the rock and lifts it out of the pit. The rope used has mass 2 kg per meter. Below, **do not simplify your final answers or evaluate any integrals**. As a reminder, the acceleration due to gravity is g , where $g = 9.8 \text{ m/s}^2$.

- a. [8 points] If the rock has mass 4 kg, write an expression involving integrals for the amount of work, in Joules, the treasure hunter does in lifting the rock and the attached rope 10 meters up from the bottom of the pit.

Hint: Once rope has been raised to the top of the pit, the treasure hunter no longer needs to lift it.

Answer: _____

- b. [8 points] After the rock has been lifted 10 meters off the bottom of the pit, the rock starts to crumble, losing 0.1 kg of mass per second. The treasure hunter resumes lifting the rock at a constant speed of 0.5 meters per second. Write an expression involving integrals for the amount of work, in Joules, the treasure hunter does in lifting the crumbling rock (and the attached rope) the remaining 5 meters to the top of the pit.

The hint from part a. still applies.

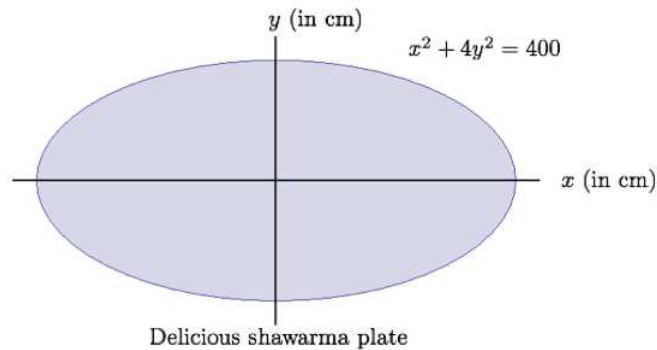
Answer: _____

9. [14 points] In a secret room at ShamCorp headquarters, there is a strangely-shaped transparent container filled with fluorescent purple liquid called “the key”. The key is in the shape of a solid with semicircular base of radius one meter, and with semicircular cross sections perpendicular to the straight side of the base. The key is suspended in the room with its semicircular cross sections parallel to the floor. The key has a volume of $\frac{\pi}{6} \text{ m}^3$, and the purple liquid has a density of 1500 kg/m^3 . The container that holds the purple liquid is infinitely thin and has no mass. For your reference, the gravitational constant is $g = 9.8 \text{ m/s}^2$.
- a. [7 points] One day, Dr. Durant orders Steph to move the key 2 meters higher. As soon as Steph begins to move the key straight up at a constant rate of 6 meters per minute, purple liquid starts leaking out of the key at a constant rate of $300\pi \text{ kg}$ per minute. Write an expression involving integrals that gives the work done by Steph moving the key 2 meters higher as it’s leaking. Do not evaluate your integral.
- b. [7 points] Periodically, Steph has to do her least favorite job — emptying the key by pumping all of the purple liquid to a height of 3 meters above the top of the key. Write an expression involving integrals that gives the work done by Steph when she does this job, assuming the key is **full** when she starts. Do not evaluate your integral.

4. [17 points]

- a. [8 points] The delicious chicken shawarma platter is served on an elliptical plate, described by the equation $x^2 + 4y^2 = 400$. The mass density of the platter, including the food, is a function of y , given by $\delta(y) = 10 + 0.5y$ grams per cm^2 .

In this problem, you do not need to evaluate any integrals.



- i) (4 points) Find an expression containing a definite integral that computes the mass of the chicken shawarma platter (including the food).

- ii) (4 points) Find expressions for the coordinates \bar{x} , \bar{y} of the center of mass of the platter. If your expression does not involve an integral, include a justification.

b. [9 points] The mouthwatering kafta kabob platter is served on a circular plate, with radius 20 cm. Including the food, the overall mass density of the platter is given by $\delta(r) = \frac{50}{2 + r^2}$ grams per cm^2 , where r is the distance from the center of the plate (in cm).

i) (4 points) Write a definite integral that computes the mass of the kafta kabob platter (including food). You do not need to evaluate the integral.

ii) (3 points) Write an estimate for your expression in part i) of the mass of the platter using LEFT(3). Show all the terms in the sum. You do not need to evaluate the sum.

iii) (2 points) Where is the center of mass of this platter? Justify.

5. [9 points] Determine whether each of the following series converges or diverges. Fully justify your answer, including carefully showing all work for any computations. Include any convergence tests used.

a. [4 points] $\sum_{n=1}^{\infty} \frac{3 - \sin(n^4)}{n^2}$

Circle one: **Converges** **Diverges**

Justification:

b. [5 points] $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$

Circle one: **Converges** **Diverges**

Justification: