MATH 116 — PRACTICE FOR EXAM 3

Generated April 19, 2026

UMID: SOLUTIONS ______ INITIALS: ______

INSTRUCTOR: ______ SECTION NUMBER: ______

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1. This exam has 15 questions. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
2. Please read the instructions for each individual exercise carefully. One of the skills being tested on this exam is your ability to interpret questions, so instructors will not answer questions about exam problems during the exam.
3. Show an appropriate amount of work (including appropriate explanation) for each exercise so that the graders can see not only the answer but also how you obtained it. Include units in your answers where appropriate.
4. You are allowed notes written on two sides of a $3'' \times 5''$ note card. You are NOT allowed other resources, including, but not limited to, notes, calculators or other electronic devices.
5. If you use graphs or tables to obtain an answer, be certain to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
6. Problems may ask for answers in exact form. Recall that $x = \sqrt{2}$ is a solution in exact form to the equation $x^2 = 2$, but $x = 1.41421356237$ is not.
7. You must use the methods learned in this course to solve all problems.

Semester	Exam	Problem	Name	Points	Score
Winter 2025	3	6		10	
Fall 2022	2	1		11	
Fall 2016	3	7	Legendre	7	
Fall 2023	3	2		8	
Winter 2011	3	9	Fourier	9	
Winter 2004	2	6	Okefenokee	12	
Fall 2025	3	4	climbing wall	13	
Winter 2025	3	9	park	12	
Winter 2024	1	2		15	
Fall 2023	3	8	vuvuzela	9	
Winter 2019	2	9	Infinity Tower	9	
Winter 2023	2	7	treasure	16	
Fall 2016	1	9	purple key	14	
Fall 2013	1	4	shawarma kafta	17	
Winter 2019	3	5		9	
Total				171	

Recommended time (based on points): 177 minutes

6. [10 points] A power series centered at $x = 3$ is given by

$$\sum_{n=1}^{\infty} \frac{2n+1}{5^n(n^2+1)}(x-3)^n.$$

The radius of convergence of this power series is 5 (do **not** show this). Find the **interval** of convergence of this power series. Show all your work, including full justification for series behavior.

Solution: Since we know the radius of convergence, we just need to test the behavior at the endpoints, which are $3 - 5 = -2$, and $3 + 5 = 8$.

At $x = -2$, the series is

$$\sum_{n=1}^{\infty} \frac{2n+1}{5^n \cdot (n^2+1)}(-5)^n = \sum_{n=1}^{\infty} \frac{(-1)^n(2n+1)}{n^2+1}.$$

To determine the behavior of this, we use the Alternating Series Test. Set $a_n = \dfrac{2n+1}{n^2+1}$. Then

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{2n+1}{n^2+1} = \lim_{n\to\infty} \frac{2n}{n^2} = \lim_{n\to\infty} \frac{2}{n} = 0.$$

and for all $n \geq 1$,

$$0 < a_{n+1} < a_n,$$

so by the Alternating Series test, $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n(2n+1)}{n^2+1}$ converges.
Therefore $x = -2$ is included in the interval of convergence.

At $x = 8$, the series is

$$\sum_{n=1}^{\infty} \frac{2n+1}{5^n \cdot (n^2+1)}(5)^n = \sum_{n=1}^{\infty} \frac{2n+1}{n^2+1}.$$

We can show that this series diverges using either the Direct Comparison Test or the Limit Comparison Test; here, we'll use the Direct Comparison Test.
For $n \geq 1$, $\dfrac{2n+1}{n^2+1} > \dfrac{2n}{n^2+n^2} = \dfrac{1}{n}$, and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, by the p-test, with $p = 1$. Therefore, by the (Direct) Comparison Test, $\displaystyle\sum_{n=1}^{\infty} \frac{2n+1}{n^2+1}$ diverges. This tells us that $x = 8$ is not included in the interval of convergence.
Therefore, the interval of convergence is $[-2, 8)$.

Interval of convergence: $[-2, 8)$

1. [11 points]

a. [7 points] Determine the **radius** of convergence of the following power series:

$$\sum_{n=1}^{\infty} \frac{9^n (x-2)^{2n}}{n^2}$$

Be sure to show all of your work. Write your final answer in the space provided below.

Solution: We use the ratio test, with $a_n = \frac{9^n(x-2)^{2n}}{n^2}$. Then:

$$\begin{aligned} \frac{|a_{n+1}|}{|a_n|} &= \frac{9^{n+1}|x-2|^{2n+2}n^2}{(n+1)^2 9^n |x-2|^{2n}} \\ &= 9|x-2|^2 \frac{n^2}{(n+1)^2} \\ &\to 9|x-2|^2 \quad \text{as } n \to \infty. \end{aligned}$$

This is less than 1 exactly when $9|x-2|^2 < 1$, or in other words $|x-2| < 1/3$. So the radius of convergence is 1/3.

Answer: 1/3.

b. [4 points] Suppose that the power series

$$\sum_{n=1}^{\infty} a_n (x-5)^n$$

converges when $x = 10$ and diverges when $x = -1$. At which of the following x-values must the series converge? Circle your answers. You do not need to show any work for this problem.

-5 0 [✓ 2] [✓ 5] 11 12

Solution: From the information given, the radius of convergence is at least 5 and at most 6. Hence it definitely converges for $|x-5|<5$, but we don't know if it converges when $|x-5|\geq 5$. So it definitely converges at $x=2,5$.

7. [7 points] The *Legendre equation* is a differential equation that arises in the quantum mechanical study of the hydrogen atom. In one of its forms, the Legendre equation is

$$(1-x^2)y'' - 2xy' + 12y = 0.$$

For this problem, let y be a solution to the Legendre equation satisfying $y(\frac{1}{2}) = 2$ and $y'(\frac{1}{2}) = 3$. Assume that the Taylor series for $y(x)$ about $x = \frac{1}{2}$ converges to $y(x)$ for all $-\frac{1}{2} < x < \frac{3}{2}$.

a. [4 points] In the blank below, write down $P_2(x)$, the degree 2 Taylor polynomial of $y(x)$ near $x = \frac{1}{2}$. Your answer should not contain the function $y(x)$ or any of its derivatives.

$$P_2(x) = \underline{\quad 2 + 3\left(x - \frac{1}{2}\right) - 14\left(x - \frac{1}{2}\right)^2 \quad}$$

b. [3 points] Compute the limit

$$\lim_{x \to 1/2} \frac{y(x) - \frac{1}{2} - 3x}{(x - \frac{1}{2})^2}.$$

Solution: The limit is -14.

2. [8 points] Consider the function $G(x) = x^3\cos(2x)$.

a. [4 points] Give the first four nonzero terms of the Taylor series of $G(x)$ centered about $x = 0$.

Solution: Using the known Taylor series of $\cos(x)$ centered at $x = 0$, we have

$$x^3\cos(2x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}(2x)^{2n}x^3 = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n}}{(2n)!}x^{2n+3}.$$

Thus, the first four nonzero terms of the Taylor series of $G(x)$ about $x = 3$ are

$$x^3 - \frac{2^2}{2!}x^5 + \frac{2^4}{4!}x^7 - \frac{2^6}{6!}x^9.$$

Answer: $x^3 - \frac{2^2}{2!}x^5 + \frac{2^4}{4!}x^7 - \frac{2^6}{6!}x^9$

b. [4 points] Find $G^{(2023)}(0)$. You do not need to simplify.

Solution: The 2023rd power of x will appear in the Taylor series expansion of $G(x)$ with a nonzero coefficient. We can see this by noting that $2n + 3 = 2023$ if $n = 1010$. Therefore,

$$\frac{G^{(2023)}(0)}{2023!} = \frac{(-1)^{1010}\,2^{2\cdot 1010}}{(2\cdot 1010)!},$$

so $G^{(2023)}(0) = (2023)! \cdot \frac{(-1)^{1010}\,2^{2020}}{2020!}$.

Answer: $(2023)! \cdot \frac{(-1)^{1010}\,2^{2020}}{2020!}$

9. [9 points]

a. [2 points] Find the Taylor series about $x = 0$ of $\sin(x^2)$. Your answer should include a formula for the general term in the series.

Solution:

$$\sin(x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+2}}{(2n+1)!} = x^2 - \frac{x^6}{3!} + \cdots + \frac{(-1)^n\, x^{4n+2}}{(2n+1)!} + \cdots$$

b. [2 points] Let m be a positive integer, find the Taylor series about $x = 0$ of $\cos(m\pi x)$. Your answer should include a formula for the general term in the series.

Solution:

$$\cos(m\pi x) = \sum_{n=0}^{\infty} \frac{(-1)^n (m\pi x)^{2n}}{(2n)!} = 1 - \frac{m^2\pi^2 x^2}{2!} + \cdots + \frac{(-1)^n (m\pi x)^{2n}}{(2n)!} + \cdots$$

c. [5 points] Use the second degree Taylor polynomials of $\sin(x^2)$ and $\cos(m\pi x)$ to approximate the value of b_m, where

$$b_m = \int_{-1}^{1} \sin(x^2)\cos(m\pi x)dx.$$

(The number b_m is called a *Fourier coefficient of the function* $\sin x^2$. These numbers play a key role in *Fourier analysis*, a subject with widespread applications in engineering and the sciences.)

Solution:

$$b_m \approx \int_{-1}^{1} x^2 \left(1 - \frac{m^2\pi^2 x^2}{2!}\right) dx$$

$$b_m \approx \int_{-1}^{1} x^2 - \frac{m^2\pi^2}{2} x^4 dx$$

$$b_m \approx \frac{x^3}{3} - \frac{m^2\pi^2}{10} x^5 |_{-1}^{1}$$

$$b_m \approx \frac{2}{3} - \frac{m^2\pi^2}{5}$$

6. (12 points) A team of biologists is interested in the ability of certain birds to migrate great distances with little rest. The biologists are monitoring a flock of birds known to migrate after spending the winter in the warm climes of the Okefenokee swamp. The location of the flock, in $x(t)$ hundreds of miles north, and $y(t)$ hundreds of miles east of the base camp of the biologists, t days after their departure from the Okefenokee swamp is given by

$$x(t) = 3t + \frac{1}{2},$$
$$y(t) = t^{\frac{3}{2}} + \frac{1}{5}.$$

(a) Where is the Okefenokee swamp in relation to the base camp of the biologists?

At $t = 0$, the birds are at the swamp, so it is located $x(0) = .5$ hundred miles north and $y(0) = .2$ hundred miles east of the base camp of the biologists. That is, 50 miles north and 20 miles east.

(b) Is there ever a time when the flock of birds is travelling due North-East? If so, when? If not, explain why not.

The flock is travelling north-east at time t if $x'(t) = y'(t) > 0$. Since $x'(t) = 3$ and $y'(t) = \frac{3}{2}t^{1/2}$, this occurs when $3 = \frac{3}{2}t^{1/2}$ or when $t = 4$ days.

(c) Is the flock of birds constantly moving throughout the first three days of their journey? Why or why not?

The velocity of the flock at time t is $v(t) = \sqrt{(x'(t))^2 + (y'(t))^2} = \sqrt{9 + \frac{9}{4}t} \geq 3 > 0$, so the flock is constantly in motion during all the days it is travelling.

(d) How far does the flock of birds travel in the first three days of their journey?

Since the rate of change of distance travelled with respect to time is the velocity, the distance travelled during the first three days is given by the integral.

$$\int_0^3 v(t)\,dt = \int_0^3 \sqrt{9 + \frac{9}{4}t}\,dt$$

One can calculate this integral, either by using the fact that an antiderivative for the integrand is $8(1 + \frac{t}{4})^{3/2}$ and applying the fundamental theorem of calculus, or by using numerical integration. The distance travelled is $(\sqrt{7})^3 - 8 \approx 10.52$ hundred miles, or about 1052 miles.

4. [13 points] A vertical climbing wall is described in coordinates, where (x, y) is the position x meters to the right of a central podium and y meters above the floor. Suppose two climbers, Sara and Tina, begin climbing the wall at the same time. Their positions t minutes after they start climbing are given by:

$$\text{Sara: } \begin{cases} x(t) = -(t-1)^2 \\ y(t) = 3t, \end{cases} \qquad \text{Tina: } \begin{cases} x(t) = 1 + \cos(\pi t) \\ y(t) = t^2 + 2. \end{cases}$$

a. [4 points] Tina and Sara bumped into each other one time during the process of climbing. At what time did this happen? Justify your answer.

Solution: They bump into each other if their x and y-coordinates are both the same. If we equate the y coordinates, we get $3t = t^2 + 2$ and solving gives us $t = 1$ and $t = 2$. Plugging in $t = 1$ and $t = 2$ to the x-coordinate parameterizations we note that the x coordinates of Sara and Tina are the same at $t = 1$. Therefore, they bumped into each other at $t = 1$.

Answer: $t =$ 1

b. [2 points] After Sara and Tina bumped into each other, they climbed for some further distance and eventually crossed the finish line, which is a horizontal line, at the same time. When did they cross the finish line, and what is the equation of the finish line?

Solution: Crossing the finish line at the same time means that Sara and Tina's y-coordinates are the same at that time. From part **a.**, we know this happens at $t = 1$ and $t = 2$. We already know from **a.** that they bumped into each other at $t = 1$, so $t = 2$ must be the time at which they both crossed the finish line. We plug in $t = 2$ to either Sara or Tina's y-coordinates and get $y = 6$.

Answer: They crossed the finish line at $t =$ 2

Answer: The equation of the finish line is $y =$ 6

c. [4 points] Write an expression involving one or more integrals that gives the total distance traveled by Tina from the time she starting climbing until she crossed the finish line. Do not evaluate your integral(s). Show your work.

Solution: For Tina, $x'(t) = -\pi \sin(\pi t)$ and $y'(t) = 2t$, so the total distance traveled is

$$\int_0^2 \sqrt{x'(t)^2 + y'(t)^2}\, dt = \int_0^2 \sqrt{\pi^2 \sin^2(\pi t) + 4t^2}\, dt.$$

Answer: $\int_0^2 \sqrt{\pi^2 \sin^2(\pi t) + 4t^2}\, dt$

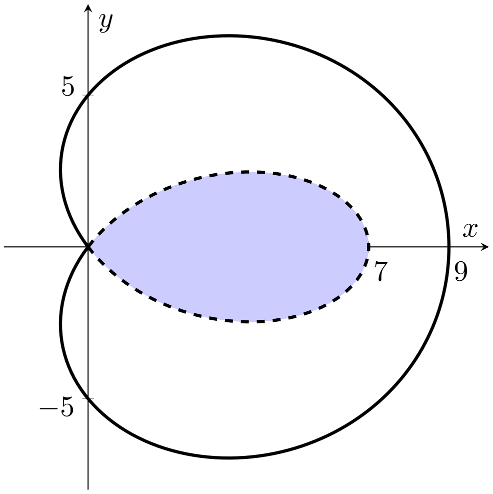
d. [3 points] What is Tina's speed at $t = \frac{1}{2}$? **Include units**. You do not need to simplify your answer but you should show your work.

Solution: Tina's speed at time t is $\sqrt{\pi^2 \sin^2(\pi t) + 4t^2}$ m/ min, so plugging in $t = \frac{1}{2}$ we get $\sqrt{\pi^2 + 1}$ m/ min .

Answer: $\sqrt{\pi^2 + 1}$ **Units:** m/min

9. [12 points]

Elena, a talented landscape architect, envisions a park whose shape is defined by the polar curve $r = 5 + 8\cos(\theta) - 4\cos^2(\theta)$, as illustrated to the right. In her design, the inner loop of the curve serves as an ideal location for a lake, represented by the shaded region. The solid outer curve in the diagram represents the walking trail that winds around the park.



a. [4 points] Using the factorization $5 + 8\cos(\theta) - 4\cos^2(\theta) = (1 + 2\cos(\theta))(5 - 2\cos(\theta))$, find the values of θ in the interval $[0, 2\pi)$ for which the curve passes through the origin.

Solution: To find the values of θ in the interval $[0, 2\pi)$ where the curve passes through the origin, we set $r = 0$. Using the factorization $5+8\cos(\theta)-4\cos^2(\theta) = (1+2\cos(\theta))(5-2\cos(\theta))$, we find that $r = 0$ when

$$\begin{aligned} 1 + 2\cos(\theta) = 0 \quad &\text{or} \quad 5 - 2\cos(\theta) = 0 \\ \cos(\theta) = -\frac{1}{2} \quad &\text{or} \quad \cos(\theta) = \frac{5}{2}. \end{aligned}$$

The equation $\cos(\theta) = -\frac{1}{2}$ has two solutions in the interval $[0, 2\pi)$: $\theta = \frac{2\pi}{3}$ and $\theta = \frac{4\pi}{3}$. The equation $\cos(\theta) = \frac{5}{2}$ has no solutions.

Answer: $\theta =$ $\frac{2\pi}{3}, \frac{4\pi}{3}$

b. [4 points] To determine the amount of water required to fill the lake, Elena wants to calculate the area of the surface of the lake. Write an expression involving one or more integrals that represents the area of the shaded region. Do not evaluate the integral(s).

Solution: From part **a**, we determine that the inner loop is traced for θ in the interval $\left(\frac{2\pi}{3}, \frac{4\pi}{3}\right)$. Using the formula for the area, the total area of the shaded region is given by

$$\int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} \frac{1}{2}\left(5 + 8\cos(\theta) - 4\cos^2(\theta)\right)^2 \, d\theta.$$

Answer: $\int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} \frac{(5 + 8\cos(\theta) - 4\cos^2(\theta))^2}{2} \, \mathbf{d}\theta$

c. [4 points] Recall that the solid outer curve in the diagram represents the walking trail. Write an expression involving one or more integrals that represents the total length of the walking trail. Do not evaluate the integral(s).

Solution: The solid outer curve is traced for θ in the intervals $\left(0, \frac{2\pi}{3}\right)$ and $\left(\frac{4\pi}{3}, 2\pi\right)$. By symmetry, both segments have equal lengths. Note that

$$\frac{\mathrm{d}r}{\mathrm{d}\theta} = -8\sin(\theta) + 8\cos(\theta)\sin(\theta)$$

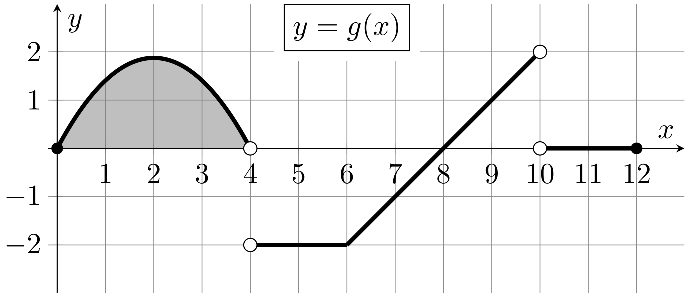
Using the formula for arc length, the total length of the walking trail is given by

$$2\int_0^{\frac{2\pi}{3}} \sqrt{\left(-8\sin(\theta) + 8\cos(\theta)\sin(\theta)\right)^2 + \left(5 + 8\cos(\theta) - 4\cos^2(\theta)\right)^2}\,\mathrm{d}\theta$$

Answer: $2\int_0^{\frac{2\pi}{3}} \sqrt{(-8\sin(\theta) + 8\cos(\theta)\sin(\theta))^2 + (5 + 8\cos(\theta) - 4\cos^2(\theta))^2}\,\mathbf{d}\theta$

2. [15 points] A function $g(x)$ is graphed below and has the following properties:

- $g(x)$ is piecewise linear for $x > 4$.
- The shaded region has area 5.



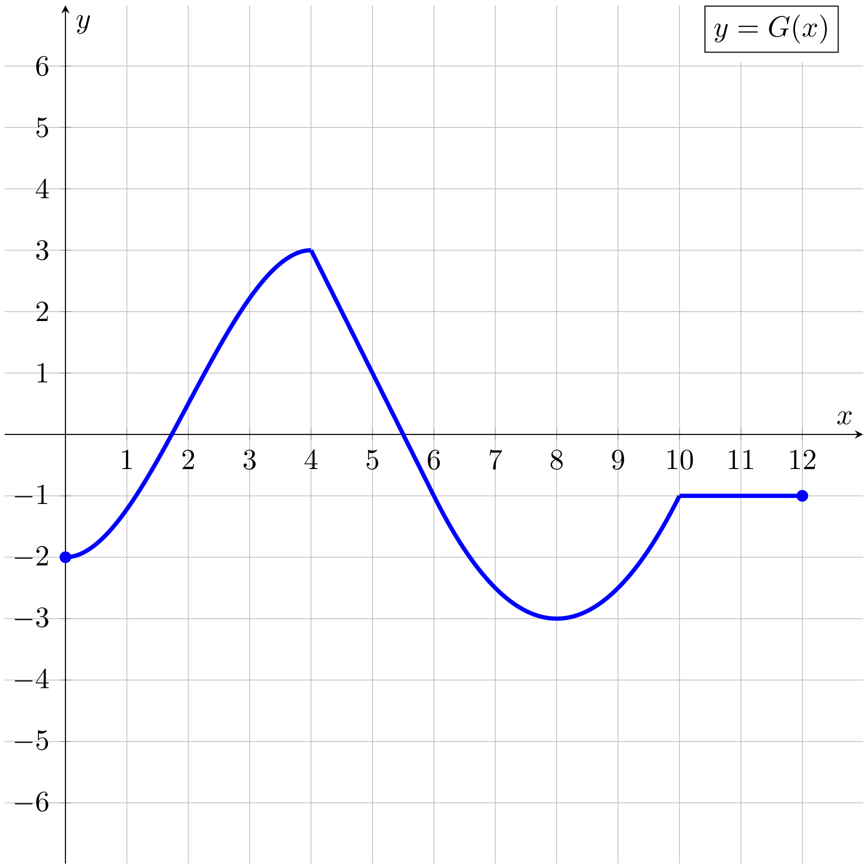
Let $G(x)$ be the continuous antiderivative of $g(x)$ satisfying $G(6) = -1$.

a. [5 points] Use the graph of $g(x)$ to complete the table below with the **exact** values of $G(x)$.

x	0	4	6	8	10	12
$G(x)$	-2	3	-1	-3	-1	-1

b. [10 points] Sketch a graph of $G(x)$ on the interval $[0, 12]$ using the axes provided below. Be sure to pay attention to:

- where $G(x)$ is and is not differentiable;
- where $G(x)$ is increasing, decreasing, or constant;
- where $G(x)$ is concave up, concave down, or linear;
- the slope of $G(x)$ at $x = 2$;
- the values of $G(x)$ you found in the table in part **a.**



Solution: The graph of $G(x)$ is above. Note the following:

The graph of $G(x)$ should be continuous on $[0, 12]$. Since $g(x)$ is defined and is continuous on $[0, 12]$ **except** at $x = 4$ and $x = 10$, then $G(x)$ must be differentiable on $(0, 12)$ **except** at $x = 4$ and $x = 10$.

Since $g(x)$ is positive on the intervals $(0, 4)$ and $(8, 10)$, then $G(x)$ should be increasing on $(0, 4)$ and $(8, 10)$.

Since $g(x)$ is negative on the interval $(4, 8)$, then $G(x)$ should be decreasing on $(4, 8)$.

Since $g(x) = 0$ on the interval $(10, 12)$, then $G(x)$ should be constant on $(10, 12)$.

Since $g(x)$ is increasing on the intervals $(0, 2)$ and $(6, 10)$, then $G(x)$ should be concave up on $(0, 2)$ and $(6, 10)$.

Since $g(x)$ is decreasing on the interval $(2, 4)$, then $G(x)$ should be concave down on $(2, 4)$.

Since $g(x)$ is constant on the intervals $(4, 6)$ and $(10, 12)$, then $G(x)$ should be linear on $(4, 6)$ and $(10, 12)$.

Since $g(2) \approx 2$, then the slope of $G(x)$ at $x = 2$ should be approximately 2.

Finally, the graph of $G(x)$ should contain the points $(0, -2)$, $(4, 3)$, $(6, -1)$, $(8, -3)$, $(10, -1)$, and $(12, -1)$.

8. [9 points] Gabriella is developing a new kind of vuvuzela. In order to come up with a new method, she first considers the old way she made her instruments.

a. [4 points] Gabriella initially made her vuvuzelas by considering a positive function $f(x)$, and forming a region $\mathcal{R}$ between $y = f(x)$ and the x-axis on the interval $[2, \infty)$. She rotated $\mathcal{R}$ about the x-axis to form the shape of the vuvuzela. Write an integral which gives the volume of the vuvuzela. Your answer will involve the function $f(x)$.

Answer: $\int_2^\infty \pi(f(x))^2 \,\mathrm{d}x$

b. [5 points] For her new batch of vuvuzelas, Gabriella considers an entirely different shape. The volume of the new design of vuvuzela is given by

$$\int_2^\infty \frac{x}{(x^2+5)^2}\,\mathrm{d}x.$$

Compute the value of this integral if it converges. If it does not converge, use a **direct computation** of the integral to show its divergence. Be sure to show your full computation, and be sure to use **proper notation**.

Solution: We begin by writing the improper integral as a corresponding limit. Using a substitution of $u = x^2 + 5$, we have

$$\begin{aligned}\int_2^\infty \frac{x}{(x^2+5)^2}\,\mathrm{d}x &= \lim_{b\to\infty}\int_2^b \frac{x}{(x^2+5)^2}\,\mathrm{d}x\\ &= \lim_{b\to\infty}\int_9^{b^2+5} \frac{(1/2)}{u^2}\,\mathrm{d}u\\ &= \lim_{b\to\infty} -\frac{1}{2(b^2+5)} + \frac{1}{2(9)}\\ &= -0 + \frac{1}{18}.\end{aligned}$$

Therefore the integral converges to $\frac{1}{18}$.

Circle one: **Diverges** [✓ **Converges to** $\frac{1}{18}$]

9. [9 points] The blueprint for the Infinity Tower has been finalized, and the design of the Tower of Hanoi is accepted. Specifically:

- the tower will have infinitely many floors
- each floor has the shape of a solid cylinder of height of 3 meters
- the nth floor has radius $\frac{1}{2n^2}$ meters
- the ground floor corresponds to $n = 1$
- the tower has constant density δ kg/m^3
- when construction begins, all materials are on the ground and have to be lifted to build each floor.

In this problem, you may assume the acceleration due to gravity is $g = 9.8$ m/s^2.

a. [7 points] Let W_n be the work, in Joules, it takes to lift the materials to build the nth floor and put that floor in place in the tower. Write an expression involving one or more integrals for each of the following.

i. $W_1 = \int_0^3 \pi \left(\frac{1}{2}\right)^2 \delta g h\, dh$

ii. $W_2 = \int_3^6 \pi \left(\frac{1}{8}\right)^2 \delta g h\, dh = \int_0^3 \pi \left(\frac{1}{8}\right)^2 \delta g (3+h)\, dh$

iii. $W_n = \int_{3(n-1)}^{3n} \pi \left(\frac{1}{2n^2}\right)^2 \delta g h\, dh = \int_0^3 \pi \left(\frac{1}{2n^2}\right)^2 \delta g (3(n-1)+h)\, dh$

b. [2 points] Write an expression involving one or more integrals and/or series that gives the total work it would take to build the entire tower. Your answer should not include the letter W.

Answer: $\sum_{n=1}^{\infty} \int_{3(n-1)}^{3n} \pi \left(\frac{1}{2n^2}\right)^2 \delta g h\, dh = \sum_{n=1}^{\infty} \int_0^3 \pi \left(\frac{1}{2n^2}\right)^2 \delta g (3(n-1)+h)\, dh$

7. [16 points] A treasure hunter has spotted a large exotic rock at the bottom of a deep pit. The vertical distance from the top of the pit to the top of the rock is 15 meters. To retrieve the rock, the treasure hunter attaches a 15 meter rope to the top of the rock and lifts it out of the pit. The rope used has mass 2 kg per meter. Below, **do <u>not</u> simplify your final answers or evaluate any integrals**. As a reminder, the acceleration due to gravity is g, where $g = 9.8\,\mathrm{m/s^2}$.

a. [8 points] If the rock has mass 4 kg, write an expression involving integrals for the amount of work, in Joules, the treasure hunter does in lifting the rock and the attached rope 10 meters up from the bottom of the pit.

Hint: Once rope has been raised to the top of the pit, the treasure hunter no longer needs to lift it.

Solution: After lifting the rock and the attached rope a distance of h meters, the length of the attached rope yet to be lifted is $15 - h$ meters. Therefore, the combined mass of the rock and the attached rope being lifted at this moment is

$$4 + 2 \cdot (15 - h) \text{ kg.}$$

So, the work done to lift the rock and the attached rope a short distance Δh meters at this moment is,

$$(4 + 2 \cdot (15 - h)) \cdot 9.8 \cdot \Delta h \text{ Joules.}$$

Therefore, the amount of work done by the treasure hunter to lift the rock and the attached rope 10 meters up from the bottom of the pit is given by,

$$\int_0^{10} (4 + 2 \cdot (15 - h)) \cdot 9.8 \, dh \text{ Joules.}$$

Answer: $\int_0^{10} (4 + 2 \cdot (15 - h)) \cdot 9.8 \, dh$

b. [8 points] After the rock has been lifted 10 meters off the bottom of the pit, the rock starts to crumble, losing 0.1 kg of mass per second. The treasure hunter resumes lifting the rock at a constant speed of 0.5 meters per second. Write an expression involving integrals for the amount of work, in Joules, the treasure hunter does in lifting the crumbling rock (and the attached rope) the remaining 5 meters to the top of the pit.

The hint from part ***a.*** *still applies.*

Solution: Since the rock loses 0.1 kg of mass per second, and the treasure hunter lifts at a constant speed of 0.5 meters per second, we have that the rock loses mass at a rate of

$$\frac{0.1\frac{\text{kg}}{\text{s}}}{0.5\frac{\text{m}}{\text{s}}} = 0.2\frac{\text{kg}}{\text{m}}.$$

Now, after lifting the rock and the attached rope a further distance of x meters, the length of the attached rope yet to be lifted is $5 - x$ meters. Therefore, the combined mass of the rock and the attached rope being lifted at this moment is

$$(4 - 0.2x) + 2 \cdot (5 - x) \text{ kg}.$$

So, the work done to lift the rock and the attached rope a short distance Δx meters at this moment is,

$$((4 - 0.2x) + 2 \cdot (5 - x)) \cdot 9.8 \cdot \Delta x \text{ Joules}.$$

Therefore, the amount of work done by the treasure hunter to lift the crumbling rock (and the attached rope) the remaining 5 meters to the top of the pit is given by,

$$\int_0^5 ((4 - 0.2x) + 2 \cdot (5 - x)) \cdot 9.8 \, dx \text{ Joules}.$$

Alternatively, we can compute quantities after we have lifted the rock and the attached rope for a duration of t seconds. The mass of the crumbling rock is $4 - 0.1t$ kg. Now, the attached rope here starts of with a mass of $2 \cdot 5 = 10$ kg, and every second 0.5 meters of it is being retracted/lifted. So, the mass of the attached rope is given by $10 - 2(0.5 \cdot t) = 10 - t$ kg. The combined mass of the rock and the attached rope is then

$$(4 - 0.1t) + (10 - t) \text{ kg}.$$

Now, in the next short period of Δt seconds, the system is lifted $0.5\Delta t$ meters. So, the work done to lift the rock and the attached rope in this short period is

$$((4 - 0.1t) + (10 - t)) \cdot 9.8 \cdot 0.5\Delta t \text{ Joules}.$$

Now, at a speed of 0.5 meters per second, it takes 10 seconds to lift the system the remaining 5 meters to the top. Therefore, the amount of work done by the treasure hunter to lift the crumbling rock (and the attached rope) in this process is given by,

$$\int_0^{10} ((4 - 0.1t) + (10 - t)) \cdot 9.8 \cdot 0.5 \, dt \text{ Joules}.$$

Answer: $\int_0^5 ((4 - 0.2x) + 2 \cdot (5 - x)) \cdot 9.8 \, dx$ **or** $\int_0^{10} ((4 - 0.1t) + (10 - t)) \cdot 9.8 \cdot 0.5 \, dt$

9. [14 points] In a secret room at ShamCorp headquarters, there is a strangely-shaped transparent container filled with fluorescent purple liquid called "the key". The key is in the shape of a solid with semicircular base of radius one meter, and with semicircular cross sections perpendicular to the straight side of the base. The key is suspended in the room with its semicircular cross sections parallel to the floor. The key has a volume of $\frac{\pi}{6}$ m^3, and the purple liquid has a density of 1500 kg/m^3. The container that holds the purple liquid is infinitely thin and has no mass. For your reference, the gravitational constant is g =9.8 m/s^2.

a. [7 points] One day, Dr. Durant orders Steph to move the key 2 meters higher. As soon as Steph begins to move the key straight up at a constant rate of 6 meters per minute, purple liquid starts leaking out of the key at a constant rate of 300π kg per minute. Write an expression involving integrals that gives the work done by Steph moving the key 2 meters higher as it's leaking. Do not evaluate your integral.

Solution: The work done is

$$g\int_0^2 (250\pi - 50\pi h)\,dh \qquad \text{J.}$$

b. [7 points] Periodically, Steph has to do her least favorite job — emptying the key by pumping all of the purple liquid to a height of 3 meters above the top of the key. Write an expression involving integrals that gives the work done by Steph when she does this job, assuming the key is **full** when she starts. Do not evaluate your integral.

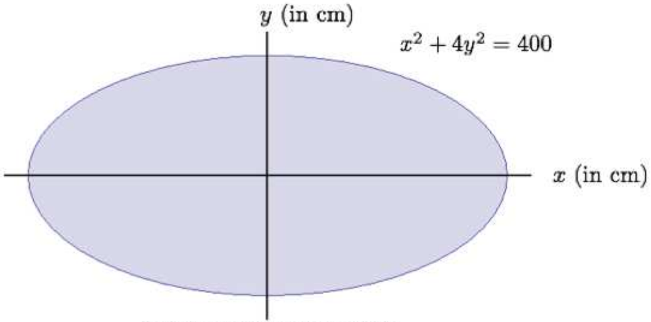
Solution: The work done is

$$\frac{1500g\pi}{8}\int_0^2 (1-(h-1)^2)(5-h)\,dh \qquad \text{J.}$$

4. [17 points]

a. [8 points] The delicious chicken shawarma platter is served on an elliptical plate, described by the equation $x^2 + 4y^2 = 400$. The mass density of the platter, including the food, is a function of y, given by $\delta(y) = 10 + 0.5y$ grams per cm^2.

In this problem, you do not need to evaluate any integrals.



Delicious shawarma plate

i) (4 points) Find an expression containing a definite integral that computes the mass of the chicken shawarma platter (including the food).

Solution:

$$m = \int_{-10}^{10} 2\sqrt{400 - 4y^2} \cdot (10 + 0.5y)dy.$$

ii) (4 points) Find expressions for the coordinates $\bar{x}$, $\bar{y}$ of the center of mass of the platter. If your expression does not involve an integral, include a justification.

Solution: $\bar{x} = 0$, since both the shape and density function are symmetric about the y-axis.

$$\bar{y} = \frac{\int_{-10}^{10} y \cdot 2\sqrt{400 - 4y^2} \cdot (10 + 0.5y)dy}{\int_{-10}^{10} 2\sqrt{400 - 4y^2} \cdot (10 + 0.5y)dy}.$$

b. [9 points] The mouthwatering kafta kabob platter is served on a circular plate, with radius 20 cm. Including the food, the overall mass density of the platter is given by $\delta(r) = \dfrac{50}{2+r^2}$ grams per cm^2, where r is the distance from the center of the plate (in cm).

i) (4 points) Write a definite integral that computes the mass of the kafta kabob platter (including food). You do not need to evaluate the integral.

Solution:

$$m = \int_0^{20} 2\pi r \cdot \frac{50}{2+r^2} dr.$$

ii) (3 points) Write an estimate for your expression in part i) of the mass of the platter using LEFT(3). Show all the terms in the sum. You do not need to evaluate the sum.

Solution:

$$\begin{aligned}
\text{LEFT}(3) &= \frac{20}{3}\left(2\pi \cdot 0 \cdot \frac{50}{2+0^2} + 2\pi \cdot \frac{20}{3} \cdot \frac{50}{2+(\frac{20}{3})^2} + 2\pi \cdot \frac{40}{3} \cdot \frac{50}{2+(\frac{40}{3})^2} \right) \\
&= \frac{20}{3}\left(0 + 45.09 + 23.30 \right). \\
&= \quad 0 + 300.6 + 155.33.
\end{aligned}$$

iii) (2 points) Where is the center of mass of this platter? Justify.

Solution: At the center of the plate, since both the shape and density function are symmetric about the origin.

5. [9 points] Determine whether each of the following series converges or diverges. Fully justify your answer, including carefully showing all work for any computations. Include any convergence tests used.

a. [4 points] $\displaystyle\sum_{n=1}^{\infty} \frac{3-\sin(n^4)}{n^2}$

Circle one: **[✓ Converges]** **Diverges**

Justification:

$$-1 \le \sin(n^4) \le 1$$

So $2 \le 3 - \sin(n^4) \le 4$

So $\dfrac{2}{n^2} \le \dfrac{3-\sin(n^4)}{n^2} \le \dfrac{4}{n^2}$

$\displaystyle\sum_{n=1}^{\infty} \frac{4}{n^2}$ converges by the p-test $(p=2)$

So since $\dfrac{3-\sin(n^4)}{n^2}$ is positive, $\displaystyle\sum \frac{3-\sin(n^4)}{n^2}$ converges by comparison.

b. [5 points] $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$

Circle one: **Converges** **[✓ Diverges]**

Justification:

Integral test:

$$\int_2^{\infty} \frac{1}{x\sqrt{\ln x}}\,dx = \lim_{b\to\infty} \int_2^{b} \frac{1}{x\sqrt{\ln x}}\,dx$$

let $w = \ln x$

$dw = \frac{1}{x}\,dx$

$x = 2 \Rightarrow w = \ln 2$

$x = b \Rightarrow w = \ln b$

$$= \lim_{b\to\infty} \int_{\ln 2}^{\ln b} \frac{1}{\sqrt{w}}\,dw = \int_{\ln 2}^{\infty} \frac{1}{w^{1/2}}\,dw$$

which diverges by the p-test $(p=\frac{1}{2})$.

So since $\dfrac{1}{n\sqrt{\ln n}}$ is positive and decreasing,

$\displaystyle\sum_{2}^{\infty} \frac{1}{n\sqrt{\ln n}}$ diverges by the integral test.