Math 115 — Practice for Exam 2

Generated November 9, 2020

UMID: **SOLUTIONS** ______ Initials: ______

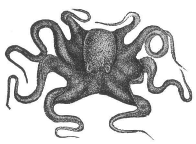
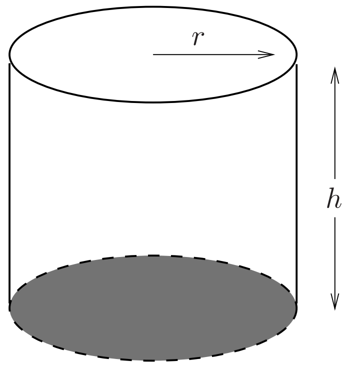
Instructor: ______________ Section Number: ______

1. This exam has 12 questions. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
2. Please read the instructions for each individual exercise carefully. One of the skills being tested on this exam is your ability to interpret questions, so instructors will not answer questions about exam problems during the exam.
3. Show an appropriate amount of work (including appropriate explanation) for each exercise so that the graders can see not only the answer but also how you obtained it. Include units in your answers where appropriate.
4. You may not use a calculator. You are allowed one double-sided 8×11 inch page of handwritten notes.
5. If you use graphs or tables to obtain an answer, be certain to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
6. You must use the methods learned in this course to solve all problems.
7. You are responsible for reading and following all directions provided for each exam in this course.

Semester	Exam	Problem	Name	Points	Score
Winter 2007	3	4	octopus2	12	
Winter 2018	2	8		14	
Fall 2017	2	2		12	
Fall 2017	2	5	Blizzard melting	8	
Winter 2017	2	6	corn snake	9	
Fall 2018	2	3		12	
Winter 2016	2	4		10	
Winter 2011	2	1	hoophouse	15	
Winter 2015	2	5		9	
Fall 2008	2	2	peak oil	12	
Fall 2018	2	2		6	
Winter 2003	2	6		10	
Total				129	

Recommended time (based on points): 120 minutes

4. (12 points) The zoo has decided to make the new octopus tank spectacular. It will be cylindrical with a round base and top. The sides will be made of Plexiglas which costs \$65.00 per square meter, and the materials for the top and bottom of the tank cost \$50.00 per square meter. If the tank must hold 45 cubic meters of water, what dimensions will minimize the cost, and what is the minimum cost?

Let V denote the volume of the tank. Then $V = \pi r^2 h = 45\ m^3$.

Solving for h gives $h = \dfrac{45}{\pi r^2}$ m.

The area of the "sides" of the tank is $2\pi r h = \dfrac{90}{r}\ m^2$, and the area of the circles for the top and bottom of the tank is $2\pi r^2$.

Thus, the cost of the tank as a function of r is

$$C(r) = 65(\frac{90}{r}) + 50(2)(\pi r^2) = \frac{5850}{r} + 100\pi r^2.$$

To minimize the cost, set $C'(r) = -\dfrac{5850}{r^2} + 200\pi r$ equal to zero, since $C'(r)$ is defined for all values of r in the domain ($r > 0$).

Solving for r, we get $r^3 = \dfrac{5850}{200\pi}$, so $r \approx 2.104$ meters.

Testing to see if this r value is in fact, the minimum, we use the second derivative test.

Since $C''(r) = 200\pi + 2\dfrac{5850}{r^3} > 0$ for all values of r in the domain, we see that C is concave up for all values of r. Since $C(r) \to \infty$ as $r \to 0$ and as $r \to \infty$, $r = 2.104$ is the global minimum of C.

radius ≈ 2.104 m

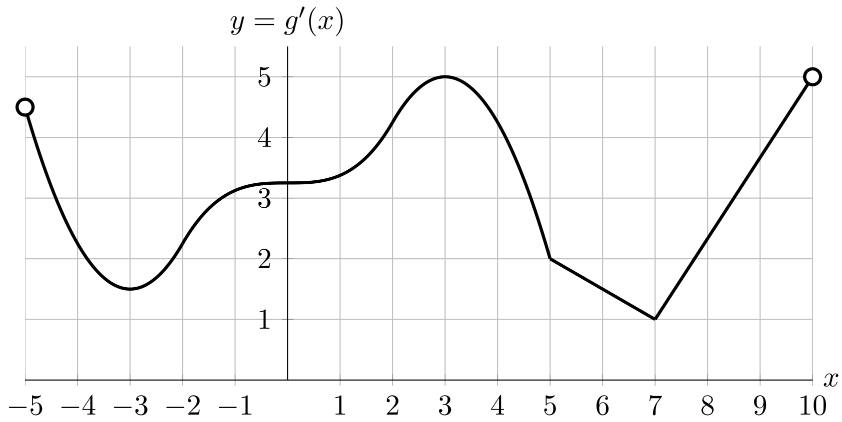
height ≈ 3.237 m

cost $\approx \$4171$

8. [14 points] The graph of the **derivative** $g'(x)$ of the function $g(x)$ with domain $-5 < x < 10$ is shown below.

The function $g'(x)$ has corners at $x = 5$ and $x = 7$, and it is linear on the intervals $(5, 7)$ and $(7, 10)$.

If there is not enough information given to answer the question, write "NEI". If the answer is none, write "None".



a. [3 points] Estimate the interval(s) on which the function $g(x)$ is concave up.

Solution:

Answer: (-3,3) and (7,10)

b. [3 points] Estimate all the x-coordinates of the inflection points of $g(x)$.

Solution:

Answer: $x = -3, 3, 7$.

c. [2 points] Estimate the values of x in $-5 < x < 10$ for which $g''(x)$ is not defined.

Solution:

Answer: $x = 5, 7$.

d. [2 points] Estimate the interval(s) on which $g'''(x) > 0$. Recall that $g'''(x)$ is the derivative of $g''(x)$.

Solution:

Answer: (approximately) $(-5, -2)$ and $(0, 1.8)$.

e. [4 points] Let $P(x)$ be the quadratic approximation of $g(x)$ at $x = 8$. Find the formula of $P(x)$ in terms of only the variable x if $g(8) = -2$. Your answer should not include the letter g.

Solution: $g(8) = -2$, $g'(8) = 1 + \frac{4}{3} = \frac{7}{3}$ and $g''(8) = \frac{4}{3}$. Then

Answer: $P(x) = -2 + \dfrac{7}{3}(x-8) + \dfrac{2}{3}(x-8)^2$

2. [12 points] Let $g(x)$ be a continuous function whose first and second derivatives are given below.

$$g'(x) = e^{2x}(2x-1)^3(x-3)^4 \qquad \text{and} \qquad g''(x) = 4e^{2x}\left(x^2-4\right)(2x-1)^2(x-3)^3$$

a. [6 points] Find all values of x at which $g(x)$ has a local extremum. Use calculus to find and justify your answers, and be sure to show enough evidence to demonstrate that you have found all local extrema. For each answer blank below, write NONE if appropriate.

Solution: The critical points of g are when $g'(x) = 0$, so at $x = 1/2$ and $x = 3$. Noticing that $e^{2x} > 0$ for all x, we see that:

- When $x < 1/2$, $g'(x) = (+)(-)(+) = (-)$, so g is decreasing.
- When $1/2 < x < 3$, $g'(x) = (+)(+)(+) = (+)$, so g is increasing.
- When $3 < x$, $g'(x) = (+)(+)(+) = (+)$, so g is increasing.

Therefore, g has a local minimum at $x = 1/2$, and no local maximum.

b. [6 points] Find all values of x at which $g(x)$ has an inflection point. Use calculus to find and justify your answers, and be sure to show enough evidence to demonstrate that you have found all inflection points. Write NONE if $g(x)$ has no points of inflection.

Solution: We see that $g''(x) = 0$ when $x = -2, 2, 1/2, 3$. We still need to check whether g changes concavity at each of these points.

- When $x < -2$, $g''(x) = (+)(+)(+)(-) = (-)$, so g is concave down.
- When $-2 < x < 1/2$, $g''(x) = (+)(-)(+)(-) = (+)$, so g is concave up.
- When $1/2 < x < 2$, $g''(x) = (+)(-)(+)(-) = (+)$, so g is concave up.
- When $2 < x < 3$, $g''(x) = (+)(+)(+)(-) = (-)$, so g is concave down.
- When $3 < x$, $g''(x) = (+)(+)(+)(+) = (+)$, so g is concave up.

Therefore g has inflection points at $x = -2, 2, 3$.

5. [8 points] Blizzard the snowman and his mouse friend Gabe arrived in Montana, where it has recently snowed. Since Blizzard is still melting, they decide to use this time to pack extra snow onto Blizzard, to help him make it to the North Pole. Let $H(t)$ be Blizzard's height, in inches, if Blizzard and Gabe stay in Montana for t hours. On the interval $1 \le t < \infty$, the function $H(t)$ can be modeled by

$$H(t) = 35 + 10e^{-t/6}(t-2)^{1/3}.$$

Notice that

$$H'(t) = \frac{-5e^{-t/6}(t-4)}{3(t-2)^{2/3}}.$$

a. [6 points] Find all values of t that give global extrema of the function $H(t)$ on the interval $1 \le t < \infty$. Use calculus to find your answers, and be sure to show enough evidence that the point(s) you find are indeed global extrema. For each answer blank, write NONE if appropriate.

Solution: The critical points of $H(t)$ occur at $t = 4$ and $t = 2$. Also checking endpoints, we have that:

- $H(1) = 26.535$
- $H(2) = 35$
- $H(4) = 41.469$
- $\lim_{t\to\infty} H(t) = 35$

and so we see that H has a global maximum when $t = 4$ and a global minimum when $t = 1$.

b. [2 points] Assuming Blizzard stays in Montana for at least 1 hour, what is the tallest height Blizzard can reach? *Remember to include units.*

Solution: Blizzard's tallest height will occur at the global maximum in the interval. Therefore, Blizzard can reach a heigh of 41.469 inches (when he's been in Montana for 4 hours).

6. [9 points] A group of biology students is studying the length L of a newborn corn snake (in cm) as a function of its weight w (in grams). That is, $L = G(w)$. A table of values of $G(w)$ is shown below.

w	5	10	15	20	25
$G(w)$	24.5	31.6	38.7	44.7	50
$G'(w)$	2.23	1.58	1.30	1.12	1.05

Assume that $G'(w)$ is a differentiable and decreasing function for $0 < w < 25$.

a. [2 points] Find a formula for $H(w)$, the tangent line approximation of $G(w)$ near $w = 20$.

Solution: The formula for is $H(w) = G(20) + G'(20)(w - 20)$. From the table we get $H(w) = 44.7 + 1.12(w - 20)$.

b. [1 point] Use the tangent line approximation of $G(w)$ near $w = 20$ to approximate the length of a corn snake that weighs 22 grams.

Solution: $G(22) \approx H(22) = 1.12(22 - 20) + 44.7 = 46.94$ cm.

c. [2 points] Is your answer in part (b) an overestimate or an underestimate? Circle your answer and write a sentence to justify it.

Solution:

Circle one: $\boxed{\text{Overestimate}}$ Underestimate CANNOT BE DETERMINED

Justification:
Since $G'(w)$ is a differentiable and decreasing function for $0 < w < 25$, then $G(w)$ is concave down on $0 < w < 25$. Hence the values of the tangent line approximation $H(w)$ will be larger than the actual values of $G(w)$ for $0 < w < 25$.

d. [4 points] In their study of the growth of corn snakes, they found the results of a recent article that states that the average weight w of a corn snake (in grams) t weeks after being born is given by $w = \frac{1}{5}t^2$. Let $S(t) = G\left(\frac{1}{5}t^2\right)$ be the length of a corn snake t weeks after being born. Find a formula for $P(t)$, the tangent line approximation of $S(t)$ near $t = 5$.

Solution: The formula for the tangent line approximation $P(t)$ is $P(t) = S(5) + S'(5)(t - 5)$. Since $S(t) = G(\frac{1}{5}t^2)$, then $S'(t) = \frac{2}{5}t \cdot G'(\frac{1}{5}t^2)$. Using these formulas we get that $S(5) = G(\frac{1}{5}(5^2)) = G(5) = 25.4$ and $S'(5) = 2 \cdot G'(5) = 4.46$.

Answer: $P(t) =$ $\underline{24.5 + 4.46(t - 5) = 4.46t + 2.2}$

3. [12 points] Assume the function $h(t)$ is invertible and $h'(t)$ is differentiable. Some of the values of the function $y = h(t)$ and its derivatives are shown in the table below

t	0	1	2	3	4
$h(t)$	-2	2	3	4	8
$h'(t)$	3.5	0.5	2.5	1.5	5
$h''(t)$	6	0.25	0.3	-0.4	0.6

Use the values in the table to compute the exact value of the following mathematical expressions. If there is not enough information provided to find the value, write NI. If the value does not exist, write DNE. **Show all your work.**

a. [3 points] Let $a(t) = h(t^2 - 5)$. Find $a'(3)$.

Solution: Since $a'(t) = 2th'(t^2 - 5)$, then $a'(3) = 6h'(4) = 6(5) = 30$

Answer: 30

b. [3 points] Let $b(t) = \dfrac{h(t)}{t^2}$. Find $b'(4)$.

Solution: Since

$$b'(t) = \frac{h'(t)t^2 - 2th(t)}{t^4} \quad \text{then} \quad b'(4) = \frac{16h'(4) - 8h(4)}{256} = \frac{16(5) - 8(8)}{256} = \frac{16}{256} = \frac{1}{16}.$$

Answer: $\frac{1}{16}$

c. [3 points] Let $c(y) = h^{-1}(y)$. Find $c'(2)$.

Solution: Since $c'(y) = \dfrac{1}{h'(h^{-1}(y))}$ then $c'(2) = \dfrac{1}{h'(h^{-1}(2))} = \dfrac{1}{h'(1)} = 2.$

Answer: 2

d. [3 points] Let $g(t) = \ln(1 + 2h'(t))$. Find $g'(0)$.

Solution: Since $g'(t) = \dfrac{2h''(t)}{1 + 2h'(t)}$ then $g'(0) = \dfrac{2h''(0)}{1 + 2h'(0)} = \dfrac{2(6)}{1 + 2(3.5)} = \dfrac{12}{8} = 1.5$

Answer: 1.5

4. [10 points] Let $h(x)$ be a twice differentiable function defined for all real numbers x. (So h is differentiable and its derivative h' is also differentiable.)
Some values of $h'(x)$, the derivative of h are given in the table below.

x	-8	-6	-4	-2	0	2	4	6	8
$h'(x)$	3	7	0	-3	-5	-4	0	-2	6

For each of the following, circle all the correct answers.
Circle "NONE OF THESE" if none of the provided choices are correct.

a. [2 points] Circle all the intervals below in which $h(x)$ must have a critical point.

$-8 < x < -6$ $\boxed{-6 < x < -2}$ $-2 < x < 2$ $\boxed{2 < x < 6}$ $\boxed{6 < x < 8}$

NONE OF THESE

b. [2 points] Circle all the intervals below in which $h(x)$ must have a local extremum (i.e. a local maximum or a local minimum).

$-8 < x < -6$ $\boxed{-6 < x < -2}$ $-2 < x < 2$ $2 < x < 6$ $\boxed{6 < x < 8}$

NONE OF THESE

c. [2 points] Circle all the intervals below in which $h(x)$ must have an inflection point.

$\boxed{-8 < x < -4}$ $-4 < x < 0$ $0 < x < 4$ $\boxed{2 < x < 6}$ $\boxed{4 < x < 8}$

NONE OF THESE

d. [2 points] Circle all the intervals below which must contain a number c such that $h''(c) = 2$.

$\boxed{-8 < x < -6}$ $-4 < x < -2$ $-2 < x < 0$ $\boxed{2 < x < 4}$ $6 < x < 8$

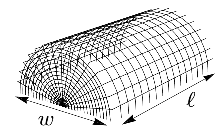
NONE OF THESE

e. [2 points] Suppose that $h''(x) < 0$ for $x < -8$, and $h(-8) = 7$. Circle all the numbers below which could equal the value of $h(-10)$.

$\boxed{-2}$ $\boxed{-1}$ $\boxed{0}$ 1 2

NONE OF THESE

1. [15 points] A hoophouse is an unheated greenhouse used to grow certain types of vegetables during the harsh Michigan winter. A typical hoophouse has a semi-cylindrical roof with a semi-circular wall on each end (see figure to the right). The growing area of the hoophouse is the rectangle of length ℓ and width w (each measured in feet) which is covered by the hoophouse. The cost of the semi-circular walls is \$0.50 per square foot and the cost of the roof, which varies with the side length ℓ, is $\$1 + 0.001\ell$ per square foot.



a. [4 points] Write an equation for the cost of a hoophouse in terms of ℓ and w. *(Hint: The surface area of a cylinder of height ℓ and radius r, not including the circles on each end, is $A = 2\pi r\ell$.)*

Solution: The roof has area $\pi r\ell = \frac{\pi}{2}w\ell$. The walls have area $\pi r^2 = \frac{\pi}{4}w^2$. This means the cost is

$$C = 0.50 \cdot \frac{\pi}{4}w^2 + (1 + 0.001\ell)\frac{\pi}{2}w\ell = \frac{\pi}{8}w^2 + \frac{\pi}{2}(1 + 0.001\ell)w\ell.$$

b. [11 points] Find the dimensions of the least expensive hoophouse with 8000 square feet of growing area.

Solution: The Area of the hoophouse is $8000 = w\ell$. Using this expression, we can eliminate ℓ in our cost equation.

$$C = \frac{\pi}{8}w^2 + \frac{\pi}{2}(1 + 0.001\ell)w\ell = \frac{\pi}{8}w^2 + \frac{\pi}{2}(1 + 0.001(8000/w))8000.$$
$$= 4000\pi + \frac{\pi}{8}w^2 + 32000\pi w^{-1}.$$

Now we compute $C' = \frac{\pi}{4}w - 32000\pi w^{-2}$. Solving for w gives us a critical point at $w =$ 50.397ft. To see what type of critical point we have, we compute $C'' = \frac{\pi}{4} + 64000\pi w^{-3}$. For $w > 0$ $C'' > 0$ which means our critical point is a local minimum by the second derivative test. Since it is the only critical point of the function, it must be a global minimum as well. When $w = 50.397$, $\ell = 158.74$, so the least expensive hoophouse with 8000 square feet of growing area is 50.397 x 158.74 ft.

5. [9 points] Consider the curve $\mathcal{C}$ defined by

$$e^{\pi x y} = a y^2 + x^2$$

where a is a positive constant.

a. [6 points] For this curve $\mathcal{C}$, find a formula for $\dfrac{dy}{dx}$ in terms of x and y. The constant a may appear in your answer. Remember to show every step of your work clearly.

Solution: We differentiate both sides of the equation defining $\mathcal{C}$ with respect to x and then solve for $\dfrac{dy}{dx}$.

$$\begin{aligned}
\frac{d}{dx}\left(e^{\pi x y}\right) &= \frac{d}{dx}\left(a y^2 + x^2\right) \\
e^{\pi x y}\frac{d}{dx}(\pi x y) &= 2ay\frac{dy}{dx} + 2x \\
\pi e^{\pi x y}\left(x\frac{dy}{dx} + y\right) &= 2ay\frac{dy}{dx} + 2x \\
\pi x e^{\pi x y}\frac{dy}{dx} + \pi y e^{\pi x y} &= 2ay\frac{dy}{dx} + 2x \\
\pi x e^{\pi x y}\frac{dy}{dx} - 2ay\frac{dy}{dx} &= 2x - \pi y e^{\pi x y} \\
\frac{dy}{dx}\left(\pi x e^{\pi x y} - 2ay\right) &= 2x - \pi y e^{\pi x y} \\
\frac{dy}{dx} &= \frac{2x - \pi y e^{\pi x y}}{\pi x e^{\pi x y} - 2ay}
\end{aligned}$$

Answer: $\dfrac{dy}{dx} = \dfrac{2x - \pi y e^{\pi x y}}{\pi x e^{\pi x y} - 2ay}$

b. [1 point] Let $a = 1$. Exactly one of the following points (x, y) lies on the curve $\mathcal{C}$. Circle that one point.

$(0, 3)$ $(1, 2)$ $(2, -1)$ $\boxed{(0, -1)}$ $(e^{\pi}, 0)$

Solution: When $a = 1$, the point $(0, -1)$ satisfies the equation defining the curve $\mathcal{C}$.

c. [2 points] With $a = 1$ as above, is the tangent line to the curve $\mathcal{C}$ at the point you chose in (b) increasing, decreasing, or is there not enough information to determine this? Circle your one choice and then justify your answer.

The tangent line to the curve $\mathcal{C}$ at the point circled in (b) is

i. $\boxed{\text{increasing.}}$ ii. decreasing. iii. NOT ENOUGH INFORMATION

Justification:

Solution: The slope of this tangent line is $\frac{dy}{dx}$ evaluated at $(0, -1)$, i.e. the slope of the tangent line is $\frac{2(0)-\pi(-1)e^{\pi(0)(-1)}}{\pi(0)e^{\pi(0)(-1)}-2(1)(-1)} = \frac{\pi}{2}$. Since $\frac{\pi}{2} > 0$, the tangent line has positive slope so is increasing.

2. In 1956, Marion Hubbert began a series of papers predicting that the United States' oil production would peak and then decline. Although he was criticized at the time, Hubbert's prediction was remarkably accurate. He modeled the annual oil production $P(t)$, in billions of barrels of oil, over time t, in years, as the *derivative* of the *logistic function* $Q(t)$ given below—*i.e.*, $Q'(t) = P(t)$. The function P is measured in years since the middle of 1910.

   The function $Q(t)$ is given by

$$Q(t) = \frac{Q_0}{1 + ae^{-bt}} \text{ , where } a, b, Q_0 > 0. \tag{1}$$

   For your convenience, the first and second derivatives of $Q(t)$ are given as well:

$$Q'(t) = -\frac{Q_0}{(1 + ae^{-bt})^2}\left(-abe^{-bt}\right) = \frac{abQ_0e^{-bt}}{(1 + ae^{-bt})^2},$$

   and

$$Q''(t) = \frac{ab^2Q_0e^{-bt}}{(1 + ae^{-bt})^3}\left[ae^{-bt} - 1\right].$$

   (a) (2 points) Interpret, in the context of this problem, $P'(56)$.

   $P'(56)$ is approximately the number of billions of barrels by which the United States' annual oil production increased from the middle of 1966 to the middle of 1967. (If $P'(56)$ is negative, then this represents a decrease in production during that time period.)

   (b) (6 points) Determine the year of maximum annual production t_{max}. Your answer may involve all or some of the constants a, b, Q_0.

   We have that $P'(t) = Q''(t) = \frac{ab^2Q_0e^{-bt}}{(1+ae^{-bt})^3}\left[ae^{-bt} - 1\right]$. The factor preceding the bracketed term is positive for all t. The bracketed term changes sign once, at $t = (1/b)\ln(a)$; so this is the only critical point of $P(t)$. The global maximimum occurs at this point, because $P'(t)$ is positive before that point and negative afterward. Thus, $t_{max} = \frac{1}{b}\ln a$.

   (c) (2 points) Find the maximum annual production $P(t_{max})$. Again, your answer may involve all or some of the constants a, b, Q_0.

   Using t_{max} from part (b), we get $P(t_{max}) = \frac{1}{4}bQ_0$.

   (d) (2 points) In his 1962 paper, Hubbert studied the available data on oil production to date and concluded that $a = 46.8$, $b = 0.0687$, and $Q_0 = 170$ Bb (billion barrels). Using your results from part (b), when would Hubbert's curve predict the peak in US oil production? (The actual peak occurred in 1964.)

   Using the results of part (b), we get $t_{max} = \frac{1}{0.0687}\ln(46.8) \approx 55.98$, which corresponds roughly to the middle of 1966.

2. [6 points] Let A and B be constants and

$$k(x) = \begin{cases} 3x + \dfrac{B}{x} & \text{for} \quad 0 < x < 1 \\ Bx^2 + Ax^3 & \text{for} \quad 1 \le x. \end{cases}$$

Find the values of A and B that make the function $k(x)$ differentiable on $(0, \infty)$. Show all your work to justify your answers. If there are no such values of A and B, write NONE.

Solution: $k(x)$ will only be differentiable at $x = 1$ if it is also continuous at $x = 1$. In order for this to happen, we plug $x = 1$ in to both formulas of the original function and set them equal as well:

$$3 + B = B + A$$

From this second equation, we can subtract B from both sides to find $A = 3$.
The function $3x + \dfrac{B}{x}$ is differentiable on $(0, 1)$, and the function $Bx^2 + Ax^3$ is differentiable on $(1, \infty)$, so we just need values of A and B that will make $k(x)$ differentiable at $x = 1$.
We can compute the derivative:

$$k'(x) = \begin{cases} 3 - \dfrac{B}{x^2} & \text{for} \quad 0 < x < 1 \\ 2Bx + 3Ax^2 & \text{for} \quad 1 < x. \end{cases}$$

In order for $k(x)$ to be differentiable at $x = 1$, we must have

$$\begin{aligned} \lim_{h \to 0^-} \frac{k(1+h) - k(1)}{h} &= \lim_{h \to 0^+} \frac{k(1+h) - k(1)}{h} \\ \frac{d}{dx}(3x + \frac{B}{x})\bigg|_{x=1} &= \frac{d}{dx}(Bx^2 + Ax^3)\bigg|_{x=1} \\ 3 - \frac{B}{x^2}\bigg|_{x=1} &= 2Bx + 3Ax^2\big|_{x=1} \\ 3 - B &= 2B + 3A \end{aligned}$$

In addition, Now we plug this value in for A in the earlier equation, giving us

$$3 - B = 2B + 9$$

Solving for B, we get $3B = -6$, so $B = -2$.

Answer: $A = 3$ $\quad B = -2$

6. (10 points) On the axes below, sketch a possible graph of a single function, $y = f(x)$, given that: [Be sure to show appropriate labels on the x axis.]

- f is defined and continuous for all real x
- f has critical points at $x = -1$ and $x = 3$
- f is decreasing for $x < 3$
- $f'(x) > 0$ for $x > 3$
- f has inflection points at $x = -1$ and $x = 1$
- f'' is positive for $x < -1$