Math 116 — Practice for Exam 1

Generated February 10, 2025

Name: SOLUTIONS

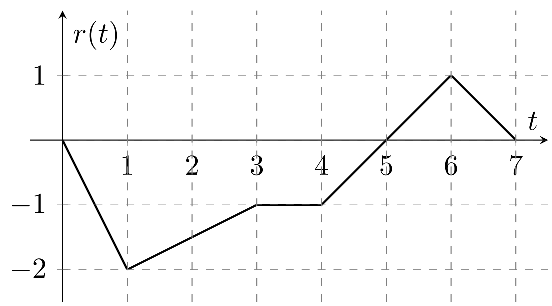
Instructor: ______________________ Section Number: __________

1. This exam has 8 questions. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
2. Do not separate the pages of the exam. If any pages do become separated, write your name on them and point them out to your instructor when you hand in the exam.
3. Please read the instructions for each individual exercise carefully. One of the skills being tested on this exam is your ability to interpret questions, so instructors will not answer questions about exam problems during the exam.
4. Show an appropriate amount of work (including appropriate explanation) for each exercise so that the graders can see not only the answer but also how you obtained it. Include units in your answers where appropriate.
5. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
6. If you use graphs or tables to obtain an answer, be certain to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
7. You must use the methods learned in this course to solve all problems.

Semester	Exam	Problem	Name	Points	Score
Fall 2020	1	2	stork	11	
Winter 2011	1	7	paint truck	11	
Fall 2020	1	8	mahogany	8	
Winter 2019	1	1		11	
Fall 2003	1	9	sandpile	12	
Fall 2019	1	9	ring	12	
Winter 2011	1	5	ramp	8	
Winter 2012	1	3	island	12	
Total				85	

Recommended time (based on points): 76 minutes

2. [11 points] In the game of *Vegetable Crossing*, Tina is carefully monitoring the stork market, which determines the price of a stork in dubloons, the game's currency. If t is the number of days since Tina started playing, then $r(t)$, measured in dubloons per day, gives the **rate of change** of the price of a stork in the game. A graph of $r(t)$ is shown below. Note that $r(t)$ is piecewise linear.



a. [2 points] For what value of t in $[0, 7]$ is the price of a stork growing fastest?

Solution: This will occur when $r(t)$ is at a maximum, so $t = 6$.

b. [2 points] Tina wants to buy storks when the price is as low as possible. For what value of t in $[0, 7]$ should she buy storks?

Solution: This will occur when the signed area between $r(t)$ and the t-axis is at a minimum, so $t = 5$.

c. [3 points] What is the average value of $r(t)$ on the interval $[3, 5]$? Be sure to write down any integrals you use to obtain your answer.

Solution: The average value of $r(t)$ on $[3, 5]$ is

$$\frac{1}{5-3}\int_3^5 r(t)dt.$$

Counting boxes using the grid, the integral has value -1.5, so the average value is -0.75.

d. [4 points] Let $R(t)$ be the price of a stork in dubloons at time t, and assume that $R(t)$ is continuous. The price of a stork at time $t = 3$ is 14 dubloons. Given that information, fill out the following table of values:

t	0	2	4	6
$R(t)$	18	$14 + \frac{5}{4}$	13	13

Solution: We get the values in the table by adding or subtracting the appropriate areas from 14, as we move toward or away from the t-value $t = 3$. For example, between $t = 2$ and $t = 3$, $R(t)$ decreases by 5/4, so $R(2) = 14 + \frac{5}{4}$.

7. [11 points]
A truck carrying a large tank of paint leaves a garage at 9AM. The tank starts to leak in such a way that x miles from the garage, the density of paint on the road is $e^{-x^2/5000}$ gallons per mile. At 10AM, a cleaning crew leaves from the same garage and follows the path of the truck, scrubbing the paint from the road as it travels until it catches up to the leaking truck. At t hours after 10AM, the leaking truck is $50\ln(t+2)$ miles from the garage, and the cleanup crew is $35t$ miles from the garage. You may use your calculator to evaluate any definite integrals for this problem.

a. [4 points] Calculate the total amount of paint that has leaked from the truck by 11AM.

Solution: The truck at 11 AM is at $50\ln 3 = 54.9306$ miles from the garage. Total amount of paint leaked from the truck is

$$\int_0^{50\ln 3} e^{-x^2/5000}dx = 45.6246 \text{ gallons.}$$

b. [2 points] At time t hours after 10AM, what interval I of the road is still covered in paint? (you may assume that t represents a time before the trucks meet)

Solution: The truck is at $50\ln(t+2)$ miles from the garage and the crew is at $35t$ miles from the garage. $I = [35t, 50\ln(t+2)]$.

c. [3 points] Let $P(t)$ represent the amount of paint in gallons on the road t hours after 10 AM. Find a formula (which may include a definite integral) for $P(t)$.

Solution:

$$P(t) = \int_{35t}^{50\ln(t+2)} e^{-x^2/5000}dx$$

d. [2 points] Calculate $P'(1)$.

Solution:

$$P'(t) = e^{-(50\ln(t+2))^2/5000}\left(\frac{50}{t+2}\right) - 35\left(e^{-(35t)^2/5000}\right)$$

$$P'(1) = e^{-(50\ln(3))^2/5000}\left(\frac{50}{3}\right) - 35\left(e^{-(35)^2/5000}\right) = -18.2795 \text{ gal/hr}$$

8. [8 points]

a. [4 points] Kesha is tasked with sourcing mahogany timber for her logging company. She finds that the cost of producing x tons of timber is given by $M(x)$, measured in thousands of dollars. She knows that $M'(x) = \ln(1 + x^2)$ and that it costs 11 thousand dollars to produce 3 tons of timber. Find an expression involving an integral for $M(x)$.

Solution:

$$M(x) = 11 + \int_3^x \ln(1 + t^2)\ dt$$

b. [4 points] Kesha finds that her department's profit for the month, in thousands of dollars, is given by

$$\int_{-5}^{5} \sin(x^5) + 2\ dx.$$

By evaluating this integral, give a numerical figure for Kesha's department's profit. (Hint: What special property of the function $\sin(x^5)$ could be useful here?) Make sure to fully justify your answer.

Solution:

$$\begin{aligned}\int_{-5}^{5} \sin(x^5) + 2\ dx &= \int_{-5}^{5} \sin(x^5)\ dx + \int_{-5}^{5} 2\ dx \\ &= \int_{-5}^{5} \sin(x^5)\ dx + 20\end{aligned}$$

But $\sin(x^5)$ is an odd function since $\sin((-x)^5) = \sin(-(x^5)) = -\sin(x^5)$ (both x^5 and $\sin(x)$ are odd). This means that $\int_{-5}^{5} \sin(x^5)\ dx = 0$, and so the profit is 20 thousand dollars.

1. [11 points] Let $f(x)$ be a differentiable function with continuous derivative, and $F(x) = \int_2^x f(t)\,dt$. Some values of the functions $f(x)$ and $F(x)$ are shown below:

x	-1	0	1	2	3
$f(x)$	7	4	0.25	9	8
$F(x)$	8	9	0.5	0	1

Compute the exact numerical values of the following integrals. If it is not possible to do so based on the information provided, write "NOT POSSIBLE" and clearly indicate why it is not possible. Show your work.

a. [3 points] $\int_0^1 x f'(x)\,dx$

Solution: By parts.

$$\int_0^1 x f'(x)\,dx = x f(x)\Big|_0^1 - \int_0^1 f(x)\,dx$$
$$= x f(x)\Big|_0^1 - F(x)\Big|_0^1$$
$$= f(1) - F(1) + F(0) = 0.25 - 0.5 + 9 = 8.75$$

Answer: $8.75 = 35/4$

b. [3 points] $\int_{-1}^0 \sqrt[3]{x} f'(x^{4/3})\,dx$

Solution: Use substitution with $u = x^{4/3}$.

$$\int_{-1}^0 \sqrt[3]{x} f'(x^{4/3})\,dx = \frac{3}{4}\int_1^0 f'(u)\,du = \frac{3}{4}\,f(u)|_1^0 = \frac{3}{4}(4 - 0.25) = \frac{45}{16} = 2.8125$$

Answer: $45/16 = 2.8125$

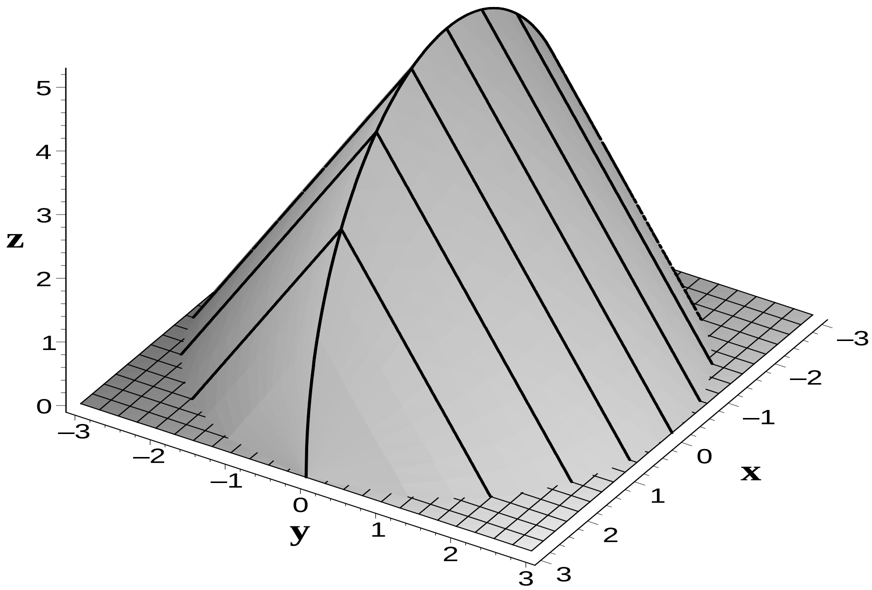
c. [5 points] $\int_1^2 \frac{f(x)}{(F(x))^2 - 1}\,dx$

Solution: Substitute $u = F(x)$ and then use partial fractions.

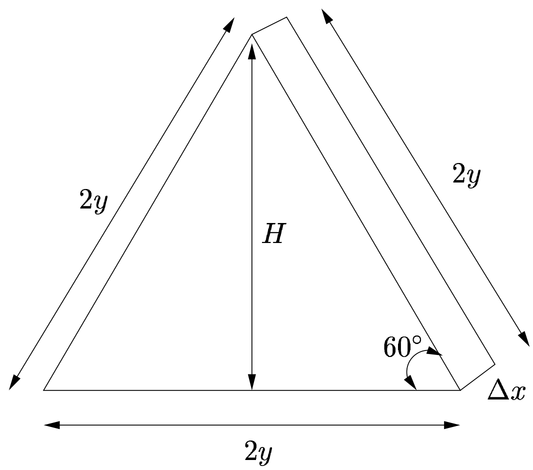
$$\int_1^2 \frac{f(x)}{(F(x))^2 - 1}\,dx = \int_{F(1)}^{F(2)} \frac{1}{u^2 - 1}\,du = \int_{0.5}^0 \frac{1/2}{u-1} - \frac{1/2}{u+1}\,du$$
$$= \frac{1}{2}\ln|u-1|\Big|_{0.5}^0 - \frac{1}{2}\ln|u+1|\Big|_{0.5}^0$$
$$= \frac{1}{2}(\ln 1 - \ln 0.5) - \frac{1}{2}(\ln 1 - \ln 1.5)$$
$$= \frac{1}{2}(\ln 1.5 - \ln 0.5) = \frac{1}{2}\ln 3 = \ln\sqrt{3}$$

Answer: $(\ln 1.5 - \ln 0.5)/2 = \ln 3/2 = \ln\sqrt{3}$

9. (12 points) It's a beautiful sunny day and you are at the beach. You manage to build the most spectacular sand castle ever. Unforunately, fate is cruel and a rogue wave hits the beach and washes over your sandcastle. But, fate also has a kinder side and it leaves you a shapely mound of sand as pictured below. The mound has as a base the interior of the circle $x^2 + y^2 = 9$ in the x-y plane and has cross sections by planes perpendicular to the x-axis given by equilateral triangles with one side in the x-y plane.



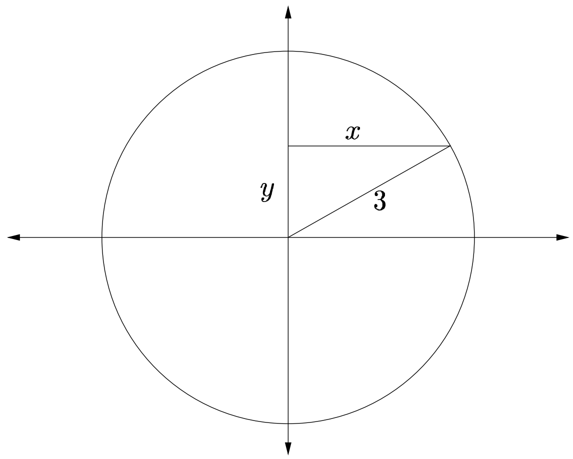
(a) Sketch and label the dimensions of a typical slice of the sand mound perpendicular to the x-axis for $-3 < x < 3$. What is the volume of this slice in terms of x?



The volume of this slice is $V_{\text{slice}} = \frac{1}{2}\,2yH\Delta x = yH\Delta x$. In order to put this in terms of x, we need to express H and y in terms of x. We can use some trigonometry to write $H = 2y\sin 60 = \sqrt{3}y$. To write y in terms of x, we use the fact that we know the base satisfies the equation $x^2 + y^2 = 9$.

From this figure we see that $y = \sqrt{9 - x^2}$. So our formula for the volume of a slice becomes

$$\begin{aligned} V_{\text{slice}} &= 2y\,H\,\Delta x \\ &= \sqrt{3}(\sqrt{9 - x^2})\,(\sqrt{9 - x^2})\,\Delta x \\ &= \sqrt{3}\,(9 - x^2)\Delta x \end{aligned}$$



(b) Write a Riemann sum and then a definite integral representing the volume of the sand pile.

Answer:

The volume of the sand pile can be approximated by adding up all the slices of volume found in part (a). This gives the Riemann sum :

$$\begin{aligned} V_{\text{sand pile}} &= \sum V_{\text{slice}} \\ &= \sum \sqrt{3}\,(9 - x^2)\,\Delta x \end{aligned}$$

Now let $\Delta x \to 0$, so the Riemann sum becomes a definite integral. The volume of the slice is then given by

$$V_{\text{sand pile}} = \sqrt{3} \int_{-3}^{3} (9 - x^2)\, dx$$

(c) Find the exact volume of the solid. If you can't compute the volume exactly, give the most accurate approximation you can and explain how you found it.

Answer:

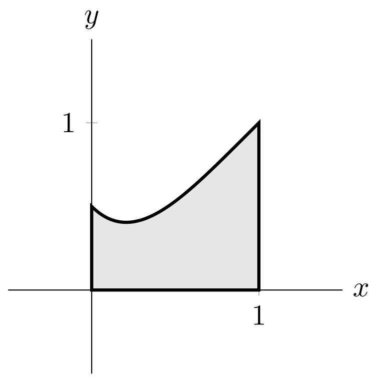
This integral is an elementary integral to evaluate, involving only power functions.

$$\begin{aligned} V_{\text{sand pile}} &= \sqrt{3} \int_{-3}^{3} (9 - x^2)\, dx \\ &= \sqrt{3} \left(9x - \frac{1}{3} x^3 \right) |_{-3}^{3} \\ &= 36\sqrt{3} \end{aligned}$$

9. [12 points] Kyle wants to make a big ring, made by the rotation of the region bounded by

$$y = x + \frac{1}{2}(x-1)^4,\ x = 0,\ x = 1, \text{ and } y = 0$$

about the line $x = -\frac{1}{2}$. This region is shown below. Both x and y are measured in centimeters.



a. [4 points] Write, but do not evaluate, an integral expression that gives the volume of Kyle's ring in cm^3.

Solution: It would be very difficult to slice with respect to y, since we would have to split the upper part into pieces. So we will slice with respect to x, which means we must use shell method.

Answer: $\displaystyle\int_0^1 2\pi\left(x+\frac{1}{2}\right)\left(x+\frac{1}{2}(x-1)^4\right)dx$

b. [4 points] The ring's density is given by $\ln(5r+1)$ grams/cm^3, where r is the distance in centimeters from the central axis of the ring. Write, but do not evaluate, an integral expressing the total mass of Kyle's ring in grams.

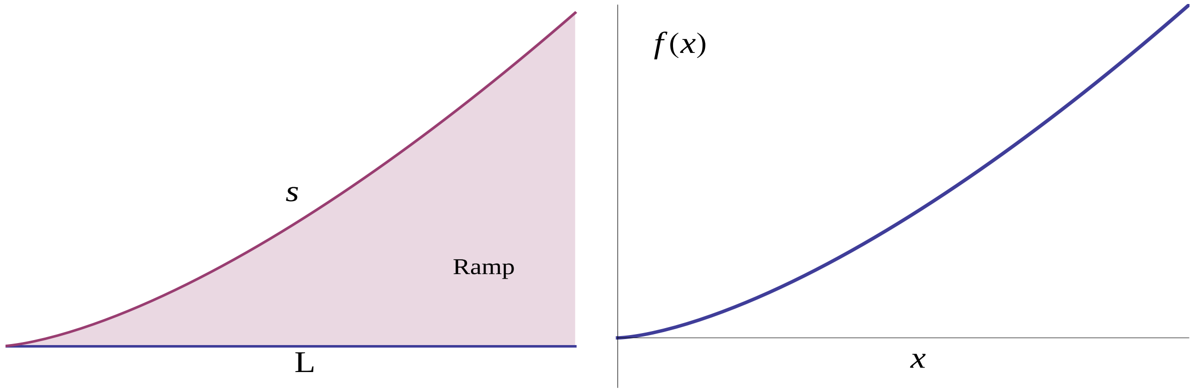
Solution: We must either write the density function in terms of the x-coordinate, or the integrand above in terms of r. Since the region is being rotated around $x = -\frac{1}{2}$, we get $r = x - -\frac{1}{2} = x + \frac{1}{2}$.

Answer: $\displaystyle\int_0^1 2\pi\left(x+\frac{1}{2}\right)\left(x+\frac{1}{2}(x-1)^4\right)\left(\ln\left(5(x+\frac{1}{2})+1\right)\right)dx$

c. [4 points] John wants to use the same region to make a ring, but instead rotates the region around the line $y = -\frac{1}{2}$.
Write, but do not evaluate, an integral that gives the **volume** of John's ring in cm^3.

Answer: $\displaystyle\int_0^1 \pi\left(\left(x+\frac{1}{2}(x-1)^4+\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2\right)dx$

5. [8 points] A company wants to design a bicycle ramp using the shape of the graph of the function $f(x) = \frac{4}{3}x^{\frac{3}{2}}$,where x is the length in meters of the base of the ramp.



Find the length s of a ramp with base of length L. Show all your work.

Solution:

$$\begin{aligned} s &= \int_0^L \sqrt{1+(f'(x))^2}dx \\ &= \int_0^L \sqrt{1+(2\sqrt{x})^2}dx \\ &= \int_0^L \sqrt{1+4x}dx \\ &= \frac{1}{6}(1+4x)^{3/2}|_0^L \\ &= \frac{1}{6}(1+4L)^{3/2} - \frac{1}{6} \end{aligned}$$

3. [12 points] A boat travels in a straight line toward an island d km away. The velocity $v(t)$ (toward the island is positive velocity) in km/hr, t hours after departing from its starting position. The velocity $v(t)$ during the first three hours is recorded at half hour intervals, and is given in the table below:

t	0	0.5	1	1.5	2	2.5	3
$v(t)$	50	48	44	38	30	20	8

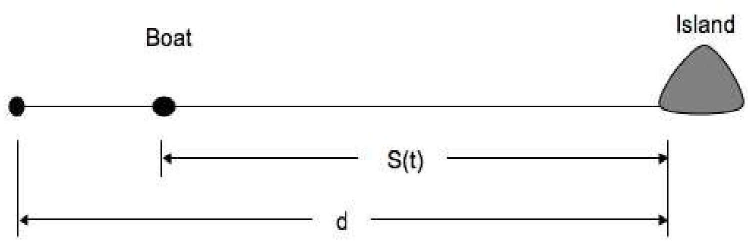
a. [8 points] Find an estimate for how far the boat is from the starting point after 3 hours using the four approximations LEFT, RIGHT, MID and TRAP. Use the maximum number of subintervals possible. Write each sum, and justify whether the sum is an underestimate or an overestimate. Assume the velocity is always decreasing and has no inflection points. **Circle your answers**.

Solution:

$$\text{LEFT}(6) = \frac{1}{2}(50+48+44+38+30+20) = 115 \qquad \text{overestimate.}$$
$$\text{RIGHT}(6) = \frac{1}{2}(48+44+38+30+20+8) = 94 \qquad \text{underestimate.}$$
$$\text{TRAP}(6) = \frac{1}{2}(115+94) = 104.5 \qquad \text{underestimate.}$$
$$\text{MID}(3) = (48+38+20) = 106 \qquad \text{overestimate.}$$

since $v(t)$ is decreasing and concave down.

b. [4 points] If it takes the boat 5 hours to reach the island, write an expression involving integrals for the distance $S(t)$ between the island and the boat for $0 \le t \le 5$.



Solution:

$$S(t) = d - \int_0^t v(x)dx.$$