Math 116 — Practice for Exam 2

Generated March 24, 2025

Name: SOLUTIONS

Instructor: ______________________ Section Number: __________

1. This exam has 10 questions. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
2. Do not separate the pages of the exam. If any pages do become separated, write your name on them and point them out to your instructor when you hand in the exam.
3. Please read the instructions for each individual exercise carefully. One of the skills being tested on this exam is your ability to interpret questions, so instructors will not answer questions about exam problems during the exam.
4. Show an appropriate amount of work (including appropriate explanation) for each exercise so that the graders can see not only the answer but also how you obtained it. Include units in your answers where appropriate.
5. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
6. If you use graphs or tables to obtain an answer, be certain to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
7. You must use the methods learned in this course to solve all problems.

Semester	Exam	Problem	Name	Points	Score
Winter 2019	2	9	Infinity Tower	9	
Fall 2013	3	2		11	
Fall 2018	1	8	pond	12	
Winter 2010	2	2		10	
Fall 2022	2	10		14	
Winter 2022	2	1	pitch speed	6	
Fall 2021	3	1	beanstalk	9	
Fall 2018	2	10		12	
Fall 2003	3	8	pendulum	12	
Fall 2005	3	5		12	
Total				107	

Recommended time (based on points): 110 minutes

9. [9 points] The blueprint for the Infinity Tower has been finalized, and the design of the Tower of Hanoi is accepted. Specifically:

- the tower will have infinitely many floors
- each floor has the shape of a solid cylinder of height of 3 meters
- the nth floor has radius $\frac{1}{2n^2}$ meters
- the ground floor corresponds to $n = 1$
- the tower has constant density δ kg/m^3
- when construction begins, all materials are on the ground and have to be lifted to build each floor.

In this problem, you may assume the acceleration due to gravity is $g = 9.8$ m/s^2.

a. [7 points] Let W_n be the work, in Joules, it takes to lift the materials to build the nth floor and put that floor in place in the tower. Write an expression involving one or more integrals for each of the following.

i. $W_1 = \int_0^3 \pi \left(\frac{1}{2}\right)^2 \delta g h \, dh$

ii. $W_2 = \int_3^6 \pi \left(\frac{1}{8}\right)^2 \delta g h \, dh = \int_0^3 \pi \left(\frac{1}{8}\right)^2 \delta g (3 + h) \, dh$

iii. $W_n = \int_{3(n-1)}^{3n} \pi \left(\frac{1}{2n^2}\right)^2 \delta g h \, dh = \int_0^3 \pi \left(\frac{1}{2n^2}\right)^2 \delta g (3(n-1) + h) \, dh$

b. [2 points] Write an expression involving one or more integrals and/or series that gives the total work it would take to build the entire tower. Your answer should not include the letter W.

Answer: $\sum_{n=1}^{\infty} \int_{3(n-1)}^{3n} \pi \left(\frac{1}{2n^2}\right)^2 \delta g h \, dh = \sum_{n=1}^{\infty} \int_0^3 \pi \left(\frac{1}{2n^2}\right)^2 \delta g (3(n-1) + h) \, dh$

2. [11 points] Determine the convergence or divergence of the following series. In parts (a) and (b), support your answers by stating and properly justifying any test(s), facts or computations you use to prove convergence or divergence. Circle your final answer. Show all your work.

a. [3 points] $\displaystyle\sum_{n=1}^{\infty} \frac{9n}{e^{-n}+n}$ CONVERGES **DIVERGES**

Solution:

$$\lim_{n\to\infty} \frac{9n}{e^{-n}+n} = \lim_{n\to\infty} \frac{9n}{n} = 9 \neq 0.$$

Since $\lim_{n\to\infty} a_n \neq 0$ then $\displaystyle\sum_{n=1}^{\infty} \frac{9n}{e^{-n}+n}$ diverges.

b. [4 points] $\displaystyle\sum_{n=2}^{\infty} \frac{4}{n(\ln n)^2}$. **CONVERGES** DIVERGES

Solution: The function $f(n) = \frac{4}{n(\ln n)^2}$ is positive and decreasing for $n > 2$, then by Integral Test the convergence or divergence of $\displaystyle\sum_{n=2}^{\infty} \frac{4}{n(\ln n)^2}$ can be determined with the convergence or divergence of $\displaystyle\int_2^{\infty} \frac{4}{x(\ln x)^2}dx$

$$\int \frac{4}{x(\ln x)^2}dx = \int \frac{4}{u^2}du \qquad \text{where } u = \ln x.$$
$$= -\frac{4}{u} + C = -\frac{4}{\ln x} + C$$

Hence

$$\int_2^{\infty} \frac{4}{x(\ln x)^2}dx = \lim_{b\to\infty} -\frac{4}{\ln x}\Big|_2^b = -\frac{4}{\ln 2} \qquad \text{converges.}$$

or

$$\int_2^{\infty} \frac{4}{x(\ln x)^2}dx = 4\int_{\ln 2}^{\infty} \frac{1}{u^2}du \qquad \text{converges by } p\text{-test with } p = 2 > 1.$$

c. [4 points] Let r be a **real** number. For which values of r is the series $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{n^2}{n^r+4}$ absolutely convergent? Conditionally convergent? No justification is required.

Solution:

Absolutely convergent if : $r > 3$

Conditionally convergent if : $2 < r \leq 3$

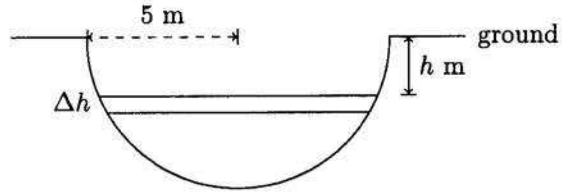
Justification (not required):

- Absolute convergence:
  The series $\displaystyle\sum_{n=1}^{\infty}\left|(-1)^n\frac{n^2}{n^r+4}\right| = \sum_{n=1}^{\infty}\frac{n^2}{n^r+4}$ behaves like $\displaystyle\sum_{n=1}^{\infty}\frac{n^2}{n^r} = \sum_{n=1}^{\infty}\frac{1}{n^{r-2}}$. The last series is a p-series with $p = r-2$ which converges if $r-2 > 1$. Hence the series converges absolutely if $r > 3$.

- Conditionally convergence:
  The function $\dfrac{n^2}{n^r+4}$ is positive and decreasing (for large values of n) when $r > 2$. Hence by the Alternating series test $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{n^2}{n^r+4}$ converges in this case.

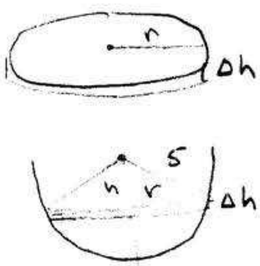
8. [12 points]

Alicia is building a pond in her backyard. The pond will be in the shape of hemisphere with radius 5 meters. A side view of the hole for the pond is shown in the figure on the right.
Note: The curved region shown is a semicircle of radius 5 meters, and cross-sections of the hole parallel to the ground are circles.



Alicia discovers that the density (in $\mathrm{kg/m^3}$) of the dirt in her yard is given by the function $\rho(h) = 1.5 + (h-1)^3$ where h is distance (in meters) below ground. In this problem, you may assume the acceleration due to gravity is $g = 9.8\ \mathrm{m/s^2}$.

a. [4 points] Write an expression that gives the approximate mass of a horizontal slice of dirt with thickness Δh meters that is h meters below the ground. See diagram. (Assume that Δh is small but positive.) Your expression should not involve any integrals.



radius of slice $= \sqrt{25-h^2}$ m

volume of slice $= \pi r^2 \Delta h = \pi(25-h^2)\Delta h$ $\mathrm{m^3}$

density of slice $= \rho(h) = 1.5 + (h-1)^3$ $\mathrm{kg/m^3}$

mass of slice $=$ (volume)(density) $=$

$r^2+h^2=5^2$

Answer: Mass of slice $\approx$ $\pi(25-h^2)(1.5+(h-1)^3)\Delta h$ kg

b. [3 points] Write, but do not evaluate, an expression involving one or more integrals that gives the mass (in kg) of the dirt Alicia removes in order to create the hole for her pond.

Answer: Mass = $\int_{h=0}^{5} \pi(25-h^2)(1.5+(h-1)^3)\,dh$ kg

c. [5 points] As Alicia digs, she lifts the dirt 1 meter above the ground to put it into the back of a truck. Write, but do not evaluate, an expression involving one or more integrals that gives the work Alicia does to remove all the dirt from the hole for her pond.

Weight of slice = (mass)(acceleration due to gravity)

$= 9.8\pi(25-h^2)(1.5+(h-1)^3)\Delta h$ N

dist to lift slice $= h+1$ m

Work to lift slice = (weight)(distance)

Answer: Work = $\int_0^5 9.8\pi(h+1)(25-h^2)(1.5+(h-1)^3)\,dh$ joules (include units)

2. [10 points] Determine if each of the following integrals diverges or converges. If the integral converges, find the exact answer. If the integral diverges, write "DIVERGES." Show ALL work and use proper notation. Calculator approximations will not receive credit.

a. [5 points] $\int_0^2 \frac{3}{x^{1/3}} dx$

Solution:

$$\begin{aligned}
\int_0^2 \frac{3}{x^{1/3}} dx &= \lim_{a\to 0^+} \int_a^2 \frac{3}{x^{1/3}} dx \\
&= \lim_{a\to 0^+} \frac{9}{2} x^{2/3}|_a^2 \\
&= \lim_{a\to 0^+} \left(\frac{9}{2}(2)^{2/3} - \frac{9}{2}a^{2/3}\right) \\
&= \frac{9}{2}(2)^{2/3}.
\end{aligned}$$

The integral converges to $\frac{9}{2}(2)^{2/3}$.

b. [5 points] $\int_0^2 \frac{e^{-1/x}}{x^2} dx$

Solution: Make the substitution $u = -\frac{1}{x}$, so $du = \frac{1}{x^2}dx$. Then we have the general antiderivative $\int \frac{e^{-1/x}}{x^2} dx = \int e^u du = e^u + C = e^{-1/x} + C$. That gives us

$$\begin{aligned}
\int_0^2 \frac{e^{-1/x}}{x^2} dx &= \lim_{a\to 0^+} \int_a^2 \frac{e^{-1/x}}{x^2} dx \\
&= \lim_{a\to 0^+} e^{-1/x}|_a^2 \\
&= \lim_{a\to 0^+} \left(e^{-1/2} - e^{-1/a}\right) \\
&= e^{-1/2}.
\end{aligned}$$

The integral converges to $e^{-1/2}$.

10. [14 points] Determine if the following series converge or diverge. Circle your final answer choice for each. Fully justify your answer including using proper notation and showing mechanics of any tests you use.

a. [7 points] $\displaystyle\sum_{n=1}^{\infty}(-1)^n \frac{n}{n^2+4n-1}$

Circle one: $\boxed{\textbf{Converges}}$ **Diverges**

Solution: With $a_n = \frac{n}{n^2+4n-1}$, we see that $a_n > a_{n+1}$ for all $n \geq 1$. Furthermore, $\lim\limits_{n\to\infty} a_n = 0$. By the alternating series test, our original series converges.

b. [7 points] $\displaystyle\sum_{n=1}^{\infty} \frac{2n-1}{n^2+n+2}$

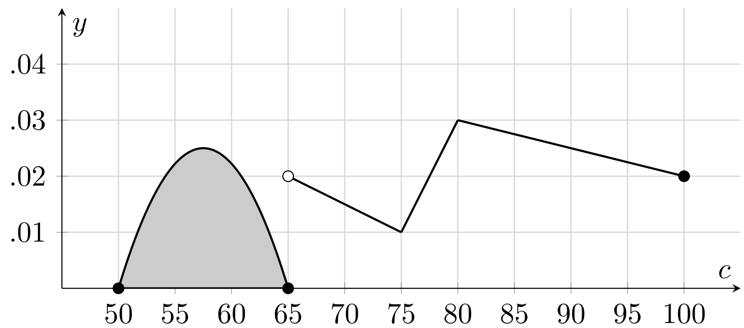
Circle one: **Converges** $\boxed{\textbf{Diverges}}$

Solution: By leading term analysis, this series has terms reminiscent of $\frac{2n}{n^2} = \frac{2}{n}$. Since $\sum_{n=1}^{\infty} \frac{2}{n}$ diverges by the p-test with $p = 1$, we expect divergence. To justify this we use LCT. Let $b_n = \frac{2}{n}$ and let $a_n = \frac{2n-1}{n^2+n+2}$. Then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2n^2-n}{2n^2+2n+4} = 1.$$

Since $0 < 1 < \infty$, LCT implies that the given series and our comparison series have the same behavior. Therefore our original series diverges.

1. [6 points] Brad and Joan have developed a new strategy to analyze baseball players, except now instead of focusing on home run distance, they need to know the probability a pitcher throws a ball at a given speed. Shown below is a graph of the function $f(c)$, a probability density function (pdf) describing the probability a certain pitcher throws the ball at a speed of c miles per hour (mph). Assume that $f(c) = 0$ when $c \leq 50$ and $c > 100$.



a. [3 points] What is the probability this pitcher throws a pitch between 50 and 65 mph?

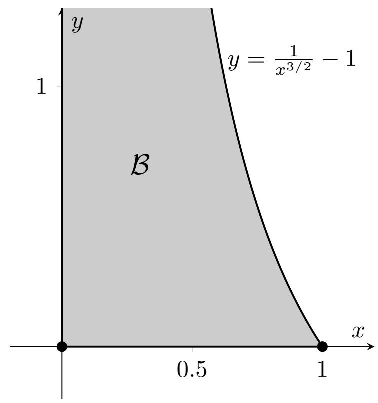
Solution: This probability is equal to $\int_{50}^{65} f(c)dc$, and so we are finding the area of the shaded region. Since the entire probability density is shown above, this area is equal to $1 - \int_{65}^{100} f(c)dc$. By breaking this area up into geometric shapes, we find that the area is 15 square units. Each unit has area equal to .05, so the total area is $15(.05) = .75$. Therefore, the final answer is $1 - .75 = .25$.

b. [3 points] What is the median speed of this player's pitches, in mph?

Solution: This can be done two ways. The first is finding M such that $\int_{50}^{M} f(c)dc = .5$. Using a), this is $.25 + \int_{65}^{M} f(c)dc = .5$, so this is equivalent finding $\int_{65}^{M} f(c)dc = .25$. Counting boxes shows that this happens at $c = 80mph$. The other way is to use the fact that $\int_{65}^{M} f(c)dc = .5$ is equivalent to $1 - \int_{M}^{100} f(c)dc = .5$. and so instead of counting boxes from left to right, we count boxes from right to left. This again gives the median as $80mph$.

1. [9 points] Arnold is building a set for his son Michael's school play in which Michael will have to climb a very tall beanstalk to fight a giant.

a. [4 points] At first, Arnold decides that since the beanstalk is extremely tall, he should model it as an infinitely tall solid of revolution of the region $\mathcal{B}$ around the y-axis. Here, $\mathcal{B}$ is the unbounded region in the first quadrant to the left of the function $f(x) = \frac{1}{x^{3/2}} - 1$ for $0 < x \leq 1$, depicted partially below.



Write an integral for just the **area** of the region $\mathcal{B}$ (and not the rotated solid) in the space below. Determine whether your integral converges or diverges, with FULL JUSTIFICATION, and circle the word CONVERGES or DIVERGES corresponding to your conclusion.

Solution: The integral is

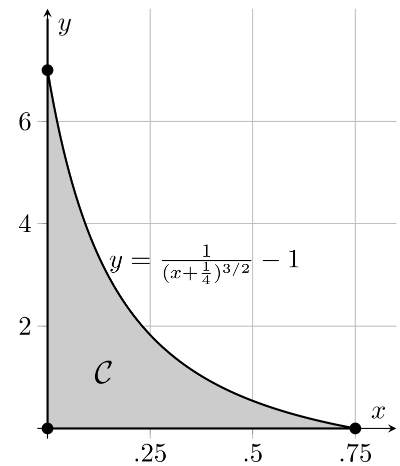
$$\int_0^1 \frac{1}{x^{3/2}} - 1dx = \int_0^1 \frac{1}{x^{3/2}}dx - 1.$$

Using the p-test ($p = \frac{3}{2}$), the integral $\int_0^1 \frac{1}{x^{3/2}}dx$ diverges, so the whole integral diverges.

The integral is $\int_0^1 \frac{1}{x^{3/2}} - 1dx$ and it CONVERGES / $\boxed{\text{DIVERGES}}$.

1. (continued)

b. [5 points] Arnold realizes modelling a beanstalk as infinitely tall is not the most realistic, so he changes his region to be $\mathcal{C}$. Here, the region $\mathcal{C}$ is bounded by the function $g(x) = \frac{1}{(x+\frac{1}{4})^{3/2}} - 1$, the x-axis, and the y-axis, depicted below.



If the model of the Beanstalk is now the solid formed by rotating the the region $\mathcal{C}$ around the y-axis, write, but do not solve, an integral that gives the **volume** of the beanstalk using the blank provided.

Solution: Taking vertical slices, we see that we obtain the shell method. The volume of a slice of thickness Δx at a horizontal coordinate x is approximately

$$\Delta V = 2\pi x \left(\frac{1}{(x+\frac{1}{4})^{3/2}} - 1 \right) \Delta x,$$

and so the total volume of the solid is

$$\int_0^{.75} 2\pi x \left(\frac{1}{(x+\frac{1}{4})^{3/2}} - 1 \right) dx.$$

Alternate Solution: Taking horizontal slices, we see that we obtain the disc method. The volume of a slice of thickness Δy at vertical coordinate y is approximately

$$\Delta V = \pi x^2 \Delta y = \pi \left((y+1)^{-2/3} - \frac{1}{4} \right)^2 \Delta y$$

and so now the total volume of the solid is

$$\int_0^7 \pi \left((y+1)^{-2/3} - \frac{1}{4} \right)^2 dy.$$

The integral is $\underline{\int_0^{.75} 2\pi x \left(\frac{1}{(x+\frac{1}{4})^{3/2}} - 1 \right) dx}$.

10. [12 points] Provide an example for each of the following. Note that there are examples in each case.

a. [3 points] A sequence a_n that is bounded but not monotonic.

Answer: $a_n =$ $(-1)^n$

b. [3 points] A sequence b_n such that $\sum_{n=1}^{\infty} b_n$ converges but $\sum_{n=1}^{\infty} b_n^2$ diverges.

Answer: $b_n =$ $\frac{(-1)^n}{\sqrt{n}}$

c. [3 points] A sequence c_n and a function $g(x)$ such that $g(n) = c_n$ for all $n \geq 1$, the improper integral $\int_1^{\infty} g(x)\,dx$ diverges, and the series $\sum_{n=1}^{\infty} c_n$ converges.

Note: You may describe your function $g(x)$ by giving either a formula or a well-drawn and clearly labeled graph.

Must violate one of the conditions of the integral test, so either not positive or not decreasing.

Answer: $c_n =$ 0 and $g(x) =$ $\sin(\pi x)$

d. [3 points] A sequence d_n with $d_n \geq 0$ for $n \geq 1$ such that

$$\lim_{n\to\infty} d_n = 0 \quad \text{and} \quad \sum_{n=1}^{\infty} (-1)^n d_n \text{ diverges.}$$

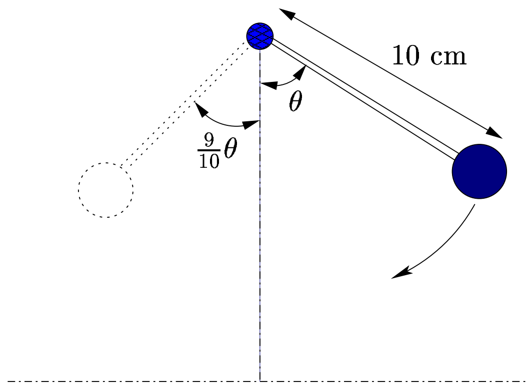
Hint: Consider defining d_n piecewise, with one formula for when n is odd and one for when n is even.

Since AST doesn't apply even though terms alternate and $\to 0$, must be the case that terms don't decrease in magnitude.

Answer: $d_n = \begin{cases} 2/n & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$

So $\sum_{1}^{\infty} (-1)^n d_n = -0 + \frac{2}{2} - 0 + \frac{2}{4} - 0 + \frac{2}{6} - \cdots = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$

8. (12 points) You begin a pendulum swinging in the position shown in the figure below with $\theta = \pi/4$. Assume the pendulum travels in a circular arc, swinging to the left past the center line shown in the figure and then returning to the right. Notice that the pendulum must briefly stop its motion before it can change direction. We define one "swing" of the pendulum to be the motion between the times when the pendulum stops its motion to change direction. For example, the first "swing" is the motion from the time you release the pendulum until it swings all the way to the left. The second "swing" is the motion coming back from the left to the right, and so on.



(a) Assume that at the end of each swing the pendulum makes an angle of $\frac{9}{10}$ the angle it made when it began the swing. What angle does the pendulum make after its second swing? After its third swing? After its n^{th} swing?

If $\theta_j > 0$ is the angle made by the pendulum with the vertical by the pendulum after the j-th swing, then $\theta_0 = \pi/4$ (given), $\theta_1 = .9\pi/4$, $\theta_2 = .9\theta_1 = (.9)^2\pi/4$, and in general $\theta_{j+1} = .9\theta_j$. Therefore,

$$\theta_2 = (.9)^2\frac{\pi}{4}, \qquad \theta_3 = (.9)^3\frac{\pi}{4}, \qquad \theta_n = (.9)^n\frac{\pi}{4}.$$

(b) Recall that the arc length of a circle is given by the formula $s = r\alpha$ where s is arc length, r is the radius of the circle, and α is the angle measuring the arc length. How far does the weight travel on its first swing? On its second swing? On its n^{th} swing?

If S_j = distance travelled on the j-th swing, then

$$\begin{aligned}
S_1 &= \textit{radius} \times \textit{angle} = 10(\theta_0 + \theta_1) = 10\left(\frac{\pi}{4} + (.9)\frac{\pi}{4}\right) = 19\frac{\pi}{4}\ \textit{cm} \\
S_2 &= 10(\theta_1 + \theta_2) = 10\left((.9)\frac{\pi}{4} + (.9)^2\frac{\pi}{4}\right) = (.9)S_1\ \textit{cm} \\
S_n &= 10(\theta_{n-1} + \theta_n) = 10\left((.9)^{n-1}\frac{\pi}{4} + (.9)^n\frac{\pi}{4}\right) = (.9)\left((.9)^{n-2}\frac{\pi}{4} + (.9)^{n-1}\frac{\pi}{4}\right) = (.9)S_{n-1} \\
&= (.9)^2 S_{n-2} = (.9)^3 S_{n-3} = \ldots (.9)^{n-1} S_1 = (.9)^{n-1}(19)\frac{\pi}{4}\ \textit{cm}
\end{aligned}$$

Problem continued on next page.

(c) What is the total distance the weight has travelled after 30 swings?

Using part (b), this number is

$$S_1 + S_2 + \dots S_{30} = S_1 + (.9)S_1 + (.9)^2 S_1 + \cdots + (.9)^{29} S_1 = S_1 \left(1 + (.9) + (.9)^2 + (.9)^{29}\right)$$
$$= S_1 \frac{1 - (.9)^{30}}{1 - .9} = 10 S_1 (1 - (.9)^{30})$$

where the next to last equality is obtained using the formula for the sum of a finite geometric series, $1 + x + x^2 + \cdots + x^{n-1} = (1 - x^n)/(1 - x)$. *Using a calculator to compute* $1 - (.9)^{30})$ *and* $S_1 = 19\frac{\pi}{4}$, *we find that the distance travelled in 30 swings is, correct to the digits shown,* $\left(190\frac{\pi}{4}\right).9576 = 142.8998$ *cm.*

(d) If the pendulum were allowed to swing forever how far would it travel?

This is similar to part (c) except that we must compute the sum of the infinite geometric series, for which we have a known formula, $1 + x + x^2 + \cdots = 1/(1 - x)$ *whenever* $|x| < 1$.

$$S_1 + S_2 + S_3 + \cdots + S_n + \dots = S_1 \left(1 + (.9) + (.9)^2 + (.9)^3 + \cdots + (.9)^n + \dots\right)$$
$$= S_1 \frac{1}{1 - .9} = 10 S_1 = 190\frac{\pi}{4} \approx 149.2257\, cm.$$

More than 95% of the total distance travelled would be taken up in the first 30 swings.

5. (12 points) Determine whether each of the following series converges or diverges. Circle CONVERGES or DIVERGES and then BRIEFLY EXPLAIN why each series converges or diverges. In each part of the problem you will receive one point for circling the correct answer (and only the correct answer) and up to two points for your explanation.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n+1}{n+2}$ $\boxed{\text{DIVERGES}}$ CONVERGES

Explanation:

Since $\displaystyle\lim_{n\to\infty} \frac{n+1}{n+2} = 1 \neq 0$, the terms of the series do not approach zero, which means that their infinite sum diverges.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{n^3}{n^5+2}$ DIVERGES $\boxed{\text{CONVERGES}}$

Explanation:

For large n, $n^3/(n^5+2) \simeq n^3/n^5 \simeq 1/n^2$. So the given series behaves like $\sum 1/n^2$, which converges by the integral test.

(c) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ $\boxed{\text{DIVERGES}}$ CONVERGES

Explanation:

Since $\int 1/(x \ln x)\, dx = \int 1/u\, du = \ln|u| + C = \ln|\ln x| + C$, and $lim_{b\to\infty} \ln|\ln b| = \infty$, the sum diverges by the integral test.

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2 2^{n+1}}{3^n}$ DIVERGES $\boxed{\text{CONVERGES}}$

Explanation:

Using the ratio test,

$$\lim_{n\to\infty} \left[\frac{(n+1)^2 2^{n+2}}{3^{n+1}} \frac{3^n}{n^2 2^{n+1}} \right] = \lim_{n\to\infty} \frac{2(n+1)^2}{3n^2} = 2/3.$$

Since $2/3 < 1$, we have convergence.