Math 116 — Practice for Exam 1

Generated February 9, 2026

UMID: **SOLUTIONS** ____________________ Initials: ______

Instructor: ______________________ Section Number: ________

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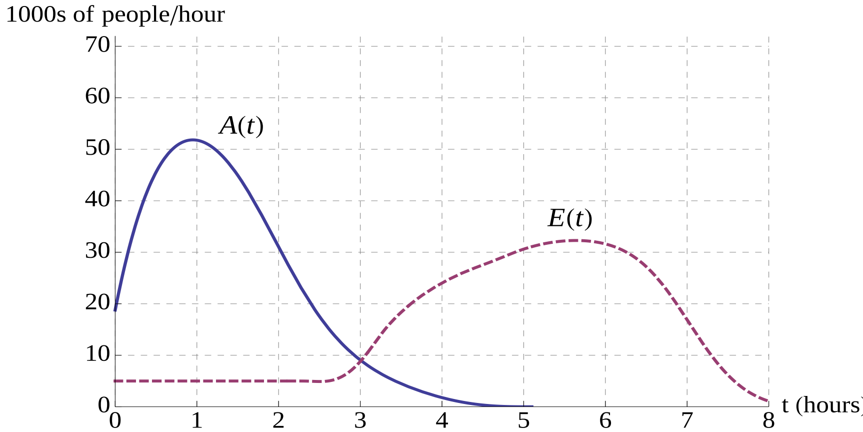
1. This exam has 10 questions. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
2. Please read the instructions for each individual exercise carefully. One of the skills being tested on this exam is your ability to interpret questions, so instructors will not answer questions about exam problems during the exam.
3. Show an appropriate amount of work (including appropriate explanation) for each exercise so that the graders can see not only the answer but also how you obtained it. Include units in your answers where appropriate.
4. You are allowed notes written on two sides of a $3'' \times 5''$ note card. You are NOT allowed other resources, including, but not limited to, notes, calculators or other electronic devices.
5. If you use graphs or tables to obtain an answer, be certain to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
6. Problems may ask for answers in exact form. Recall that $x = \sqrt{2}$ is a solution in exact form to the equation $x^2 = 2$, but $x = 1.41421356237$ is not.
7. You must use the methods learned in this course to solve all problems.

Semester	Exam	Problem	Name	Points	Score
Fall 2013	1	2	Big House	12	
Winter 2017	1	2	plankton	13	
Fall 2022	1	1		16	
Winter 2024	1	5		14	
Winter 2015	1	4		14	
Fall 2003	1	9	sandpile	12	
Winter 2019	1	6	smoothies	12	
Winter 2014	1	11	table leg	5	
Fall 2016	1	3	fishtank	11	
Fall 2019	1	8	park	11	
Total				120	

Recommended time (based on points): 108 minutes

2. [12 points] The following is a graph showing the rates at which visitors arrive at and exit from the Big House during a football game:



The solid line $A(t)$ shows the arrival rate and the dashed line $E(t)$ shows the exit rate, in thousands of people per hour. Time t is measured in hours after 10 a.m. At 10 am, there are already 10,500 people inside the stadium.

a. [5 points] Let $P(t)$, the number of people, in thousands, in the stadium at time t. Give an expression for $P(t)$.

Solution:

$$P(t) = 10.5 + \int_0^t A(x) - E(x)dx.$$

b. [3 points] Give a practical interpretation of the statement $\int_1^4 A(t)dt = 67$. Include units.

Solution: 67,000 people arrived at the Big House between 11 a.m. and 2 p.m.

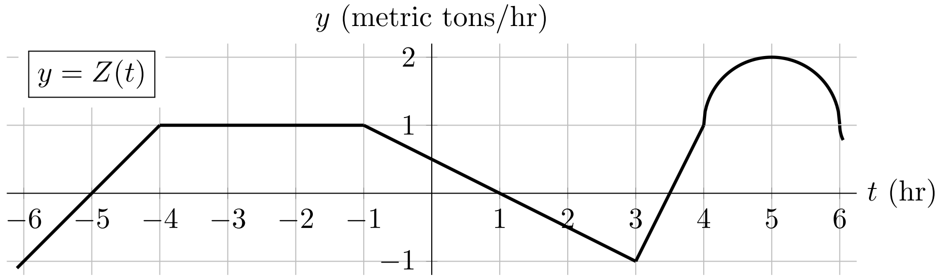
c. [2 points] At what time(s) of the day are there the most people in the stadium?

Solution: $t = 3$ (or 1 p.m).

d. [2 points] To comply with safety regulations, the game organizers must determine an upper bound (overestimate) of the number of people in the stadium at the beginning of the game. Circle the estimation rule(s) they could use. The game begins at noon.

left right trap **MID**

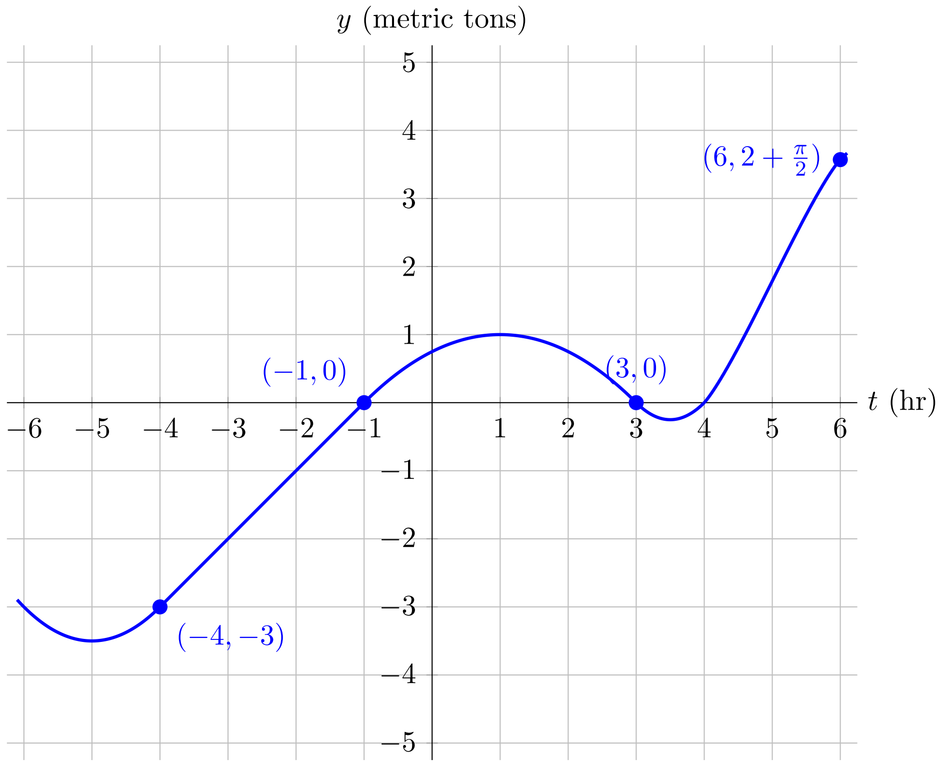
2. [13 points] Suppose $Z(t)$ is the rate of change, in metric tons per hour, of the biomass (i.e. total mass) of zooplankton in Loch Ness t hours after 8am on January 25, 2017.
Below is a portion of the graph of $Z(t)$. Note that this graph is linear on the intervals $[-6,-4]$, $[-4,-1]$, $[-1,3]$, and $[3,4]$. Also note that the portion of the graph for $4 \le t \le 6$ is the upper half of a circle centered at the point $(5,1)$.



Let $B(t)$ be the biomass, in metric tons, of zooplankton in Loch Ness t hours after 8am on January 25, 2017.

a. [10 points] Carefully sketch a graph of $y = B(t) - B(3)$ for $-6 \le t \le 6$ using the axes provided below. If there are features of this function that are difficult for you to draw, indicate these on your graph. Be sure that local extrema and concavity are clear.
Label the coordinates of the points on your graph at $t = -4, -1, 3, 6$.

Solution: Note that $B(t) - B(3) = \int_3^t Z(x)\,dx$ is the antiderivative of $Z(t)$ whose value is 0 at $t = 3$. Although it is difficult to tell here, the graph below is concave up for $4 < t < 5$ and concave down for $5 < t < 6$.



b. [3 points] Define $A(h)$ to be the average biomass (in metric tons) of zooplankton in Loch Ness during the first h hours after 8am on January 25, 2017. Write an expression for $A(h)$. (Your expression may involve integrals, the function Z, and/or the function B.)

Solution: $A(h)$ is the average value of the function $B(t)$ over the interval $0 \le t \le h$, so

$$A(h) = \frac{1}{h}\int_0^h B(t)\,dt.$$

1. [16 points] Use the table to compute the following quantities. The function $h(x)$ is **odd**, **twice differentiable**, and $h'(x) > 0$ for all x-values. Write your answer using exact form on the blank provided. If there is not enough information available to answer the question, write N.I. You need to evaluate all integrals, but you do not need to simplify your final answer.

x	0	1	2	3	4
$h(x)$	0	2	4	5	7
$h'(x)$	7	3	4	7	2

a. [4 points] $\displaystyle\int_3^4 \frac{h''(t)}{h'(t)}\,dt$

Answer: $\ln(2/7)$.

Solution: Let $u = h'(t)$, so $du = h''(t)dt$ and the integral becomes $\displaystyle\int_7^2 u^{-1}\,du = \ln(2) - \ln(7)$ which is $\ln(2/7)$.

b. [4 points] The average value of $h'(x)$ on $[-1, 1]$

Answer: **2**.

Solution: The average value of $h'(x)$ on $[-1, 1]$ is by definition $\displaystyle\frac{1}{1-(-1)}\int_{-1}^{1} h'(x)\,dx$. Using the fundamental theorem of calculus, this is equal to $\frac{1}{2}(h(1) - h(-1))$. Since $h(x)$ is odd, $h(-1) = -h(1) = -2$, so the answer is $\frac{1}{2}(2 - (-2)) = 2$.

c. [4 points] $\displaystyle\int_1^4 (w+1)h''(w)\,dw$

Answer: -1.

Solution: Since integration splits over addition, this is equal to $\displaystyle\int_1^4 wh''(w)\,dw + \int_1^4 h''(w)\,dw$. For the first integral we integrate by parts to obtain $\displaystyle\int_1^4 wh''(w)\,dw = -\int_1^4 h'(w)\,dw + wh'(w)|_1^4 = h(1) - h(4) + 4h'(4) - h'(1) = 0$. For the second integral we use FTC to get $\displaystyle\int_1^4 h''(w)\,dw = h'(4) - h'(1) = 2 - 3 = -1$.

d. [4 points] $\displaystyle\int_{1/2}^{2} x^{-1/2}h'(\sqrt{2x})\,dx$

Answer: $2\sqrt{2}$.

Solution: Let $u = \sqrt{2x} = (2x)^{1/2}$. By the chain rule and power rule, $du = \frac{1}{\sqrt{2x}}dx$. Therefore $dx = \sqrt{2x}du$. So the integral given is equal to $\sqrt{2}\int_1^2 h'(u)\,du = \sqrt{2}(h(2) - h(1)) = 2\sqrt{2}$.

5. [14 points]

a. [6 points] Split the following expression into partial fractions with two or more terms. **Do not integrate these terms.** Please clearly show all of your work.

$$\frac{5x-4}{(x-2)^2(x+1)}$$

Solution: The partial fraction decomposition of the given function has the form

$$\frac{5x-4}{(x-2)^2(x+1)} = \frac{A}{x-2} + \frac{B}{(x-2)^2} + \frac{C}{x+1}.$$

By multiplying to obtain a common denominator, we have

$$5x-4 = A(x-2)(x+1) + B(x+1) + C(x-2)^2. \qquad (*)$$

Below are two possible ways to complete the problem:

Solution 1 (Comparing coefficients)**:** By multiplying out the terms on the right-hand side of the equation (*) and grouping terms with the same powers of x, we obtain

$$0x^2 + 5x - 4 = (A+C)x^2 + (-A+B-4C)x + (-2A+B+4C).$$

This gives the system of equations

$$A+C=0, \qquad -A+B-4C=5, \qquad -2A+B+4C=-4.$$

The first equation implies $C=-A$, so the second and third equations become

$$3A+B=5, \qquad -6A+B=-4.$$

Taking this first equation and subtracting the second equation from it, we obtain $9A=9$. Thus $A=1$, so $3A+B=5$ implies $B=2$, and $C=-A$ implies $C=-1$. This gives us the answer.

Solution 2 (Plugging in values)**:** By setting $x=2$ in the equation (*), we obtain

$$5(2)-4 = A(2-2)(2+1) + B(2+1) + C(2-2)^2.$$

This simplifies to $6 = 0 + 3B + 0$, thus $B=2$. Now setting $x=-1$ in (*), we obtain

$$5(-1)-4 = A(-1-2)(-1+1) + B(-1+1) + C(-1-2)^2.$$

This simplifies to $-9 = 0 + 0 + 9C$, thus $C=-1$. Finally, setting $x=0$ in (*) and using the facts that $B=2$ and $C=-1$, we obtain

$$5(0)-4 = A(0-2)(0+1) + 2(0+1) + (-1)(0-2)^2.$$

This simplifies to $-4 = -2A + 2 - 4$, thus $A=1$. This gives us the answer.

Answer: $\dfrac{1}{x-2} + \dfrac{2}{(x-2)^2} + \dfrac{-1}{x+1}$

b. [8 points] Use the partial fraction decomposition

$$\frac{4x-2}{(3-x)(x^2+1)} = \frac{1}{3-x} + \frac{x-1}{x^2+1}$$

to evaluate the following **indefinite** integral. Please clearly show all of your work.

$$\int \frac{4x-2}{(3-x)(x^2+1)}\,dx.$$

Solution: Using the partial fraction decomposition:

$$\int \frac{4x-2}{(3-x)(x^2+1)}\,dx = \int \frac{1}{3-x}\,dx + \int \frac{x-1}{x^2+1}\,dx.$$

Now we use linearity:

$$\int \frac{4x-2}{(3-x)(x^2+1)}\,dx = \int \frac{1}{3-x}\,dx + \int \frac{x}{x^2+1}\,dx - \int \frac{1}{x^2+1}\,dx.$$

The first integral is a standard antiderivative (or, substitute $u = 3-x$):

$$\int \frac{1}{3-x}\,dx = -\ln|3-x| + C.$$

The second integral can be evaluated using the substitution $u = x^2+1$, so that $du = 2x\,dx$, and thus

$$\int \frac{x}{x^2+1}\,dx = \frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln|x^2+1| + C.$$

The third integral is a standard antiderivative:

$$-\int \frac{1}{x^2+1}\,dx = -\arctan(x) + C.$$

Putting this all together, our final answer is

$$\int \frac{4x-2}{(3-x)(x^2+1)}\,dx = -\ln|3-x| + \frac{1}{2}\ln|x^2+1| - \arctan(x) + C.$$

Answer: $-\ln|3-x| + \frac{1}{2}\ln|x^2+1| - \arctan(x) + C$

4. [14 points] The function

$$f(x) = \sin(\sqrt{x})$$

does not have an antiderivative that can be written in terms of elementary functions. However, we can use the second fundamental theorem of calculus to construct an antiderivative for f. We define an antiderivative F of f by

$$F(x) = \int_0^x \sin(\sqrt{t})dt.$$

a. [2 points] The concavity of F does not change on the interval $\left(0, \frac{\pi^2}{4}\right)$. Determine the concavity of F on $\left(0, \frac{\pi^2}{4}\right)$ and circle one of the options below. No justification is needed.

Solution:

$\boxed{\textbf{Concave Up}}$ **Concave Down** **Neither**

b. [2 points] Using the blanks provided, order from least to greatest

$$F\left(\frac{\pi^2}{4}\right), \quad \text{LEFT}(100), \quad \text{RIGHT}(100), \quad \text{MID}(100), \quad \text{TRAP}(100),$$

where all the approximations are of the definite integral given by $F\left(\frac{\pi^2}{4}\right)$. No justification is needed.

Solution: LEFT (100) $\leq$ TRAP (100) $\leq$ $F\left(\frac{\pi^2}{4}\right)$ $\leq$ MID (100) $\leq$ RIGHT (100)

c. [4 points] Write out, but do not compute, MID (3) to approximate $F\left(\frac{\pi^2}{4}\right)$.

Solution: $\text{MID}(3) = \left[\sin\left(\sqrt{\frac{\pi^2}{24}}\right) + \sin\left(\sqrt{\frac{3\pi^2}{24}}\right) + \sin\left(\sqrt{\frac{5\pi^2}{24}}\right)\right]\left(\frac{\pi^2}{12}\right)$

d. [4 points] Write out, but do not compute, TRAP (3) to approximate $F\left(\frac{\pi^2}{4}\right)$.

Solution: We have $\text{LEFT}(3) = \left(\sin(0) + \sin\left(\sqrt{\frac{\pi^2}{12}}\right) + \sin\left(\sqrt{\frac{2\pi^2}{12}}\right)\right)\left(\frac{\pi^2}{12}\right)$,
and $\text{RIGHT}(3) = \left(\sin\left(\sqrt{\frac{\pi^2}{12}}\right) + \sin\left(\sqrt{\frac{2\pi^2}{12}}\right) + \sin\left(\sqrt{\frac{3\pi^2}{12}}\right)\right)\left(\frac{\pi^2}{12}\right)$.
Then $\text{TRAP}(3) = \frac{\text{LEFT}(3)+\text{RIGHT}(3)}{2}$.

e. [2 points] If you want to approximate $F\left(\frac{\pi^2}{4}\right)$ using right and left sums, what is the smallest number of subdivisions, n, you would have to use to guarantee that the difference between LEFT(n) and RIGHT(n) is less than or equal to 0.005?

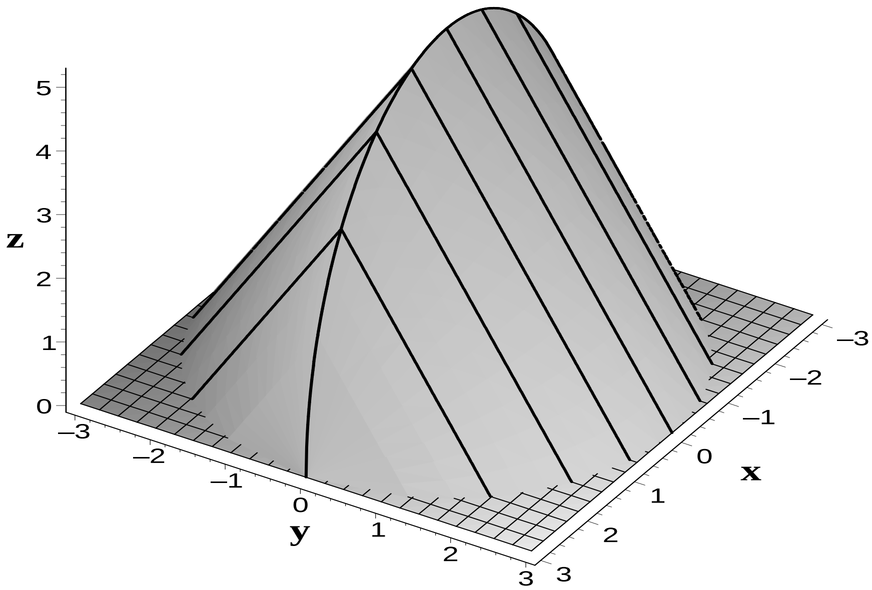
Solution:

$$\text{RIGHT}\,(n) - \text{LEFT}\,(n) < 0.005 \iff \left(\sin\left(\sqrt{\frac{\pi^2}{4}}\right) - \sin(0)\right)\frac{\frac{\pi^2}{4} - 0}{n} < 0.005$$

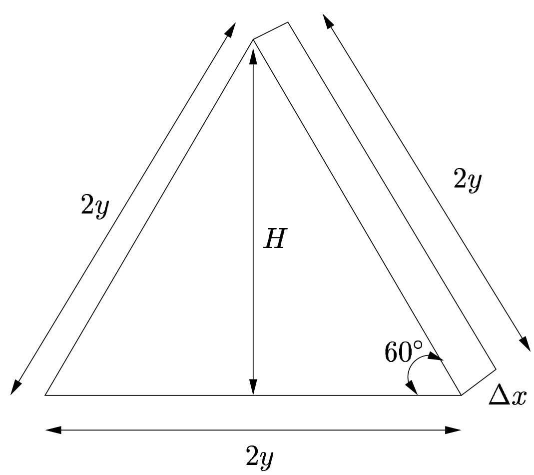
$$\iff n > \frac{\pi^2}{0.02} \approx 493.$$

Since we need an integer number of subdivisions, we take $n = 494$.

9. (12 points) It's a beautiful sunny day and you are at the beach. You manage to build the most spectacular sand castle ever. Unforunately, fate is cruel and a rogue wave hits the beach and washes over your sandcastle. But, fate also has a kinder side and it leaves you a shapely mound of sand as pictured below. The mound has as a base the interior of the circle $x^2 + y^2 = 9$ in the x-y plane and has cross sections by planes perpendicular to the x-axis given by equilateral triangles with one side in the x-y plane.



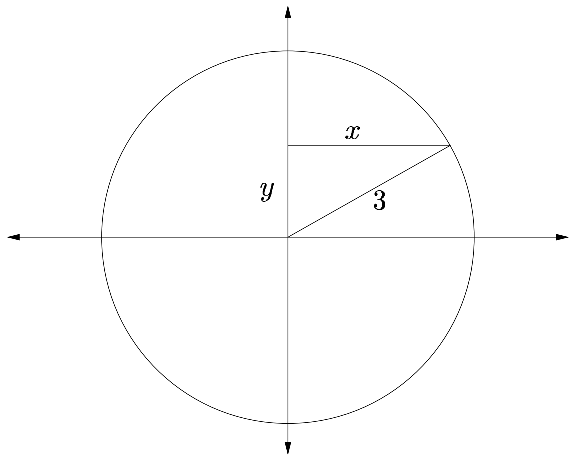
(a) Sketch and label the dimensions of a typical slice of the sand mound perpendicular to the x-axis for $-3 < x < 3$. What is the volume of this slice in terms of x?



The volume of this slice is $V_{\text{slice}} = \frac{1}{2}\, 2yH\Delta x = yH\Delta x$. In order to put this in terms of x, we need to express H and y in terms of x. We can use some trigonometry to write $H = 2y \sin 60 = \sqrt{3}y$. To write y in terms of x, we use the fact that we know the base satisfies the equation $x^2 + y^2 = 9$.

From this figure we see that $y = \sqrt{9 - x^2}$. So our formula for the volume of a slice becomes

$$
\begin{aligned}
V_{\text{slice}} &= 2y\, H\, \Delta x \\
&= \sqrt{3}(\sqrt{9 - x^2})\,(\sqrt{9 - x^2})\,\Delta x \\
&= \sqrt{3}\,(9 - x^2)\Delta x
\end{aligned}
$$



(b) Write a Riemann sum and then a definite integral representing the volume of the sand pile.

Answer:
The volume of the sand pile can be approximated by adding up all the slices of volume found in part (a). This gives the Riemann sum :

$$\begin{aligned} V_{\text{sand pile}} &= \sum V_{\text{slice}} \\ &= \sum \sqrt{3}\,(9 - x^2)\,\Delta x \end{aligned}$$

Now let $\Delta x \to 0$, so the Riemann sum becomes a definite integral. The volume of the slice is then given by

$$V_{\text{sand pile}} = \sqrt{3}\int_{-3}^{3}(9 - x^2)\,dx$$

(c) Find the exact volume of the solid. If you can't compute the volume exactly, give the most accurate approximation you can and explain how you found it.

Answer:
This integral is an elementary integral to evaluate, involving only power functions.

$$\begin{aligned} V_{\text{sand pile}} &= \sqrt{3}\int_{-3}^{3}(9 - x^2)\,dx \\ &= \sqrt{3}\left(9x - \frac{1}{3}x^3\right)\Big|_{-3}^{3} \\ &= 36\sqrt{3} \end{aligned}$$

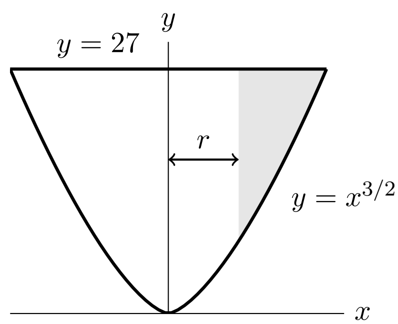
6. [12 points] Ryan Rabbitt is making a smoothie with his new electric drink mixer. Mathematically, the container of the mixer has a shape that can be modeled as the surface obtained by rotating the region in the first quadrant bounded by the curves $y = 27$ and $y = x^{3/2}$ about the y-axis, where all lengths are measured in centimeters.

a. [7 points] Write, but do not evaluate, two integrals representing the total volume, in cm^3, the mixer can hold: one with respect to x, and one with respect to y.

Answer (with respect to x): $\int_0^9 2\pi x\left(27 - x^{3/2}\right)\,dx$

Answer (with respect to y): $\int_0^{27} \pi\left(y^{2/3}\right)^2\,dy$

b. [5 points] Ryan adds 1600 cubic centimeters of liquid to his mixer. The container spins around the y-axis at a very high speed, causing the liquid to move away from the center of the container. The result is the solid made by rotating the shaded region around the y-axis in the diagram below. Note that this means that there is an empty space inside the liquid that has the shape of a cylinder.



Let r be the radius of this cylinder of empty space. Set up an equation involving one or more integrals that you would use to solve to find the value of r. **Do <u>not</u> solve for r.**

Solution:

$$\int_r^9 2\pi x\left(27 - x^{3/2}\right)\,dx = 1600,$$

or

$$\int_{r^{3/2}}^{27} \pi\left(y^{2/3}\right)^2\,dy - \pi r^2(27 - r^{3/2}) = 1600.$$

(There are other equations that would also work.)

Answer:

10. [6 points] Suppose that $p(x)$ and $q(x)$ are functions defined on $[2,4]$ with $0 \leq p(x) < q(x) \leq 3$ for all x in $[2,4]$. Let R be the region enclosed by the graphs of $p(x), q(x)$ and the lines $x = 2$ and $x = 4$.
For the following questions circle the correct answer. You do not need to show work.

a. [3 points] What is the volume of the solid obtained by rotating R about the line $y = 5$?

A: $\int_2^4 \pi[25 - (p(x) - q(x))^2]dx$

$\boxed{\text{B: } \int_2^4 \pi[(5 - p(x))^2 - (5 - q(x))^2]dx}$

C: $\int_2^4 \pi[(5 - (q(x) - p(x)))^2]dx$

D: $\int_2^4 \pi[(5 - q(x))^2 - (5 - p(x))^2]dx$

b. [3 points] What is the volume of the solid obtained by rotating R about the line $x = 7$?

$\boxed{\text{A: } \int_2^4 2\pi(7 - x)(q(x) - p(x))dx}$

B: $\int_2^4 2\pi x(q(x) - p(x))dx$

C: $\int_2^4 2\pi(7 + x)(q(x) - p(x))dx$

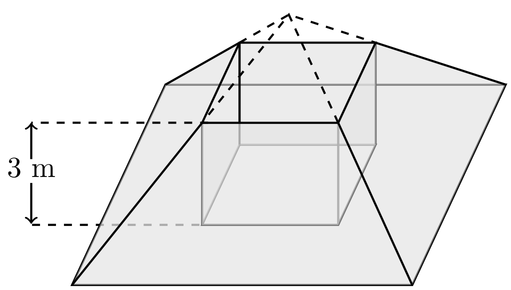
D: $\int_2^4 2\pi(x - 7)(q(x) - p(x))dx$

11. [5 points] A giant table leg is being built by rotating the region bounded by the graph of $y = \frac{1}{2}\cos(2\pi x) + 2$, the x-axis, the line $x = 0$, and the line $x = 1$ about the x-axis. Assume the units of x and y are in meters. Write an integral which gives the volume of the table leg. Do not evaluate the integral. What are the units of this integral?

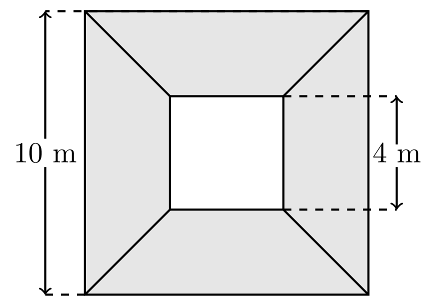
The volume of the table leg is given by the integral $\int_0^1 \pi(\frac{1}{2}\cos(2\pi x)+2)^2 dx$

The units of this integral are m^3

3. [11 points] During a trip to the local aquarium, Steph becomes curious and decides to taste the fish food. The fish food tank is completely filled with food, and it is in the shape of a pyramid with a vertical hole through its center, illustrated below (the dashed lines are not part of the tank). The tank itself is 3 m tall, and the pyramid base is a square of side length 10 m. The top and bottom of the hole are squares of side length 4 m. The food is contained in the shaded region only, **not** in the hole.



SIDE VIEW



TOP VIEW

a. [5 points] Write an expression that gives the approximate volume of a slice of fish food of thickness Δh meters, h meters from the bottom of the tank.

Solution: The approximate volume is

$$((10-2h)^2 - 4^2)\Delta h \qquad \mathrm{m}^3.$$

b. [3 points] Suppose that the mass density of fish food is a constant δ kg/m^3. Write, but do **not** evaluate, an expression involving integrals that gives the mass of fish food in the tank.

Solution: The mass of fish food in the tank is given by

$$\delta \int_0^3 ((10-2h)^2 - 4^2)\, dh \qquad \mathrm{kg}.$$

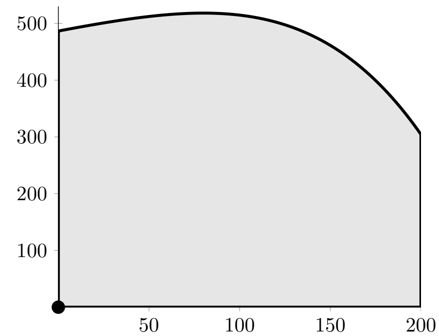
c. [3 points] Write an expression involving integrals that gives $\overline{h}$, the h-coordinate of the center of mass of the fish food, where h is defined as above. Do **not** evaluate your expression.

Solution: We have

$$\overline{h} = \frac{\int_0^3 h((10-2h)^2 - 4^2)\, dh}{\int_0^3 ((10-2h)^2 - 4^2)\, dh} \qquad \mathrm{m}.$$

8. [11 points]

A city is designing a park, which will have three straight sides and one curved side, as shown in the diagram at right. There is a fountain at the southwest corner of the park, located at $(0, 0)$. Let $p(x)$ be the north-south distance, in feet, across the park x feet to the east of the fountain. Note that the park is 200 feet wide in the east-west direction.



City planners have the following data for $p(x)$.

x	0	50	100	150	200
$p(x)$	487	512	515	462	305
$p'(x)$	0.606	0.364	-0.373	-1.91	-4.57

a. [6 points] The planners would like to be able to say the following:

The area of the park is at most ________ square feet.

Given this, should they use an overestimate or an underestimate to approximate the area of the park?

Answer *(circle one)*: $\boxed{\text{OVERESTIMATE}}$ UNDERESTIMATE

Which one of the following methods of approximation should they use to guarantee this?

RIGHT(n) LEFT(n) $\boxed{\text{MID}(n)}$ TRAP(n)

Find the approximation you chose above, using the maximal amount of equal subintervals possible, for the area of the park in square feet. Write out all the terms in your sum.

Answer: $\text{MID}(2) = 100(p(50) + p(150)) = 100(512 + 462) = 97400$

b. [5 points] They plan to put a fence along the entire northern (curved) side. Write an expression involving one or more integrals that gives the total length of this fence in feet.

Answer: $\displaystyle\int_0^{200} \sqrt{1 + (p'(x))^2}\, dx$

Use a RIGHT(2) approximation to estimate the length of the fence. Write out all the terms in your sum.

Answer: $100(\sqrt{1 + (-0.373)^2} + \sqrt{1 + (-4.57)^2}) = 574.543$