MATH 116 — PRACTICE FOR EXAM 2

Generated March 23, 2026

UMID: **SOLUTIONS** ____________________ INITIALS: ______

INSTRUCTOR: ____________________ SECTION NUMBER: ________

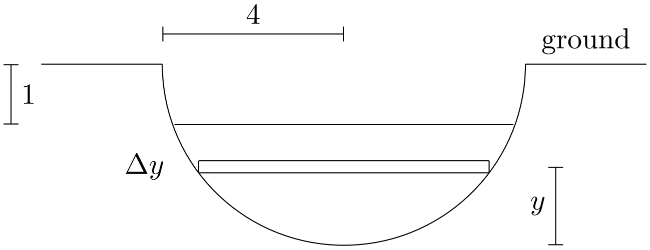
---

1. This exam has 9 questions. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
2. Please read the instructions for each individual exercise carefully. One of the skills being tested on this exam is your ability to interpret questions, so instructors will not answer questions about exam problems during the exam.
3. Show an appropriate amount of work (including appropriate explanation) for each exercise so that the graders can see not only the answer but also how you obtained it. Include units in your answers where appropriate.
4. You are allowed notes written on two sides of a $3'' \times 5''$ note card. You are NOT allowed other resources, including, but not limited to, notes, calculators or other electronic devices.
5. If you use graphs or tables to obtain an answer, be certain to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
6. Problems may ask for answers in exact form. Recall that $x = \sqrt{2}$ is a solution in exact form to the equation $x^2 = 2$, but $x = 1.41421356237$ is not.
7. You must use the methods learned in this course to solve all problems.

Semester	Exam	Problem	Name	Points	Score
Fall 2024	1	9	bird bath	8	
Winter 2025	3	8	box	6	
Winter 2022	2	7		12	
Winter 2021	2	7		6	
Winter 2024	2	6	song writing	10	
Winter 2022	3	4		8	
Fall 2024	2	9	sand tower	12	
Winter 2025	2	11		12	
Fall 2021	2	6		14	
Total				88	

Recommended time (based on points): 83 minutes

9. [8 points] Marcy's gigantic bird bath attracts lots of birds to her garden. The bath is carved out of the ground, in the shape of a **hemisphere** with radius 4 meters. A cross-section of the bath is depicted below. The bath is partially filled with muddy water, so that the surface of the water is 1 meter below ground level. The density of the water in the bath is given by the function $\delta(y)$ (measured in kilograms per cubic meter), where y is measured in meters from the **bottom of the bath**. You may assume that the acceleration due to gravity is $g = 9.8\text{m/s}^2$.



a. [4 points] Consider a horizontal slice of muddy water, y meters from the bottom of the bath with a small thickness of Δy meters, as depicted in the diagram above. Write an expression which approximates the mass, in kilograms, of this slice as a function of y. Your answer may include $\delta(y)$. Your answer should **not** involve any integrals.

Solution: Let r be the radius of the slice at height y. Using the Pythagorean Theorem, $r = \sqrt{16 - (4-y)^2}$. Therefore, the mass of the slice is

$$\delta(y)\pi r^2 \Delta y = \delta(y)\pi \left(16 - (4-y)^2\right) \Delta y = \delta(y)\pi \left(8y - y^2\right) \Delta y$$

Answer: $\delta(y)\pi \left(16 - (4-y)^2\right) \Delta y = \delta(y)\pi \left(8y - y^2\right) \Delta y$

b. [4 points] Write an expression involving one or more integrals that gives the work done, in joules, to pump all the water in the bath up to ground level. Do not evaluate your integral(s).

Solution: Using our expression for the mass of a slice from part a., we see that the weight of a slice, in newtons, is:

$$\delta(y)\pi g \left(8y - y^2\right) \Delta y$$

A horizontal slice at height y will need to be lifted a distance of $4 - y$ meters. Hence the work done to move such a slice is

$$\left(\delta(y)\pi g \left(8y - y^2\right) g\Delta y\right)(4-y) = \delta(y)\pi g \left(8y - y^2\right)(4-y)\Delta y$$

The water fills the bath up to height 3 meters, so the total work done to pump all the water in the bath up to ground level is

$$\int_0^3 \delta(y)\pi g \left(8y - y^2\right)(4-y)\, dy$$

Answer: $\int_0^3 \delta(y)\pi g \left(8y - y^2\right)(4-y)\, dy$

8. [6 points] A team of miners is working to extract a box of minerals from a deep pit. The box weighs 40 lbs, and the rope used to lift it weighs 3 lbs per foot. Initially, when the box is at the bottom of the pit, the rope is 60 feet long. As the box is lifted, the miners do not need to lift the portion of the rope that has already been "reeled in", that is, the part that has reached the top of the pit.

a. [3 points] At a certain moment, the box has already been lifted h feet above the ground. Find an expression for the total weight, in pounds, of the box together with the attached rope that has not yet been reeled in.

Solution: We denote by $F(h)$ the total weight, in pounds, of the box together with the portion of rope that has not yet been reeled in at the instant when the box is h feet above the ground. Because the rope was originally 60 feet long, the length still hanging is $60-h$ feet. Multiplying this length by the rope's weight density and then adding the weight of the box gives

$$F(h) = 3(60-h) + 40.$$

Answer: $3(60-h)+40$

b. [3 points] Using your expression from part (a), find an expression involving one or more integrals that represents the total work done on the box and attached rope, in foot-pounds, to lift the box from the base of the pit to a point 35 feet above its original position. Do not evaluate any integrals that appear in your answer.

Solution: Suppose the box has already been lifted to a height h feet above the ground. The work required to lift it an additional Δh feet is given by:

$$F(h)\,\Delta h = (3(60-h)+40)\,\Delta h.$$

Therefore, the total work required to lift the box from the bottom of the pit to a point 35 feet above its initial position is given by

$$W = \int_0^{35} (3(60-h)+40)\ \mathrm{d}h.$$

Answer: $\int_0^{35} (3(60-h)+40)\ \mathbf{d}h$

7. [12 points] The parts of this problem are unrelated to each other.

a. [7 points] **Compute** the value of the following improper integral if it converges. If it does not converge, use a **direct computation** of the integral to show its divergence. Be sure to show your full computation, and be sure to use **proper notation.**

$$\int_1^2 \frac{1}{\sqrt{t-1}}dt$$

Solution: First, this is an improper integral at $t = 1$. Therefore, we need to switch to limit notation:

$$\int_1^2 \frac{1}{\sqrt{t-1}}dt = \lim_{b\to 1^+} \int_b^2 \frac{1}{\sqrt{t-1}}dt$$

Now, we do a u-sub, with $u = t - 1$, so $du = dt$, so our integral becomes

$$\lim_{b\to 1^+} \int_{b-1}^1 \frac{1}{\sqrt{u}}du = \lim_{b\to 1^+} 2\sqrt{u}\Big|_{b-1}^1$$

Evaluating we get:

$$\lim_{b\to 1^+} (2\sqrt{1} - 2\sqrt{b-1}) = 2$$

b. [5 points] Compute the following limit. Fully justify your answer including using proper notation.

$$\lim_{x \to 0} \quad \frac{1 - \cos(x)}{x^2}$$

Solution: As x goes to zero, this becomes an indeterminant form of $\frac{0}{0}$, so we apply L'Hopital to get

$$\lim_{x \to 0} \quad \frac{1 - \cos(x)}{x^2} = \lim_{x \to 0} \frac{\sin(x)}{2x}.$$

This is also an indeterminant form, so we use L'Hopital again, to get:

$$\lim_{x \to 0} \frac{\sin(x)}{2x} = \lim_{x \to 0} \frac{\cos(x)}{2}.$$

Computing the final limit gives

$$\lim_{x \to 0} \frac{\cos(x)}{2} = \frac{\cos(0)}{2} = \frac{1}{2}.$$

So the final answer is

$$\lim_{x \to 0} \quad \frac{1 - \cos(x)}{x^2} = \frac{1}{2}$$

7. [6 points] Determine whether the following improper integral converges or diverges. **Fully justify** your answer including using **proper notation**, and showing mechanics of any tests or theorems you use.

$$\int_0^1 \frac{\pi}{x^3+\sqrt{x}}\,dx$$

Solution: This integral is improper at the lower bound 0, since the denominator is 0 at $x=0$. Consider the dominating terms in the denominator. As $x \to 0^+$, $\sqrt{x}$ dominates x^3. As a result we should compare the integrand to $\pi/\sqrt{x}$.

In any case, since both x^3 and $\sqrt{x}$ are positive, if we take x^3 away, the denominator gets smaller. Hence the fraction gets bigger. Thus,

$$0 < \frac{\pi}{x^3+\sqrt{x}} \leq \frac{\pi}{\sqrt{x}} \qquad \text{for } 0 < x \leq 1.$$

The integral

$$\int_0^1 \frac{\pi}{\sqrt{x}}\,dx$$

converges by p-test, $p = 1/2$. Therefore, by comparison test,

$$\int_0^1 \frac{\pi}{x^3+\sqrt{x}}\,dx$$

converges.

6. [10 points] Liban is writing songs using a new style of music which he calls "new-age jazz." The longer that he spends writing a particular song, the better it turns out.

Let $Q(t)$ be the **cumulative distribution function** (cdf) for t, the number of days that it takes for Liban to write a particular song. The formula for $Q(t)$ is shown to the right, where $c > 0$ is a constant.

$$Q(t) = \begin{cases} 0 & t < 0, \\ \frac{c}{4}t^2 & 0 \le t \le 2, \\ 2c - c\,e^{2-t} & t > 2. \end{cases}$$

You do not need to show your work in this problem, but partial credit may be given for work shown.

a. [3 points] Write a piecewise-defined formula for $q(t)$, the **probability density function** (pdf) corresponding to $Q(t)$. Your answer may involve c, but it should not involve the letter Q.

Solution: We know that $Q(t)$ and $q(t)$ are related by the formula $Q'(t) = q(t)$. So, the formula for $q(t)$ is found by taking the derivative of each part of $Q(t)$.

$$q(t) = \begin{cases} \underline{\quad 0 \quad} & t < 0, \\ \underline{\quad \frac{c}{2}t \quad} & 0 \le t \le 2, \\ \underline{\quad c\,e^{2-t} \quad} & t > 2. \end{cases}$$

b. [3 points] Write an expression involving one or more integrals that represents the **mean** number of days that it takes for Liban to write a particular song. Your answer may involve c, but it should not involve the letters Q or q. **Do not evaluate your integral(s).**

Solution: The formula for the mean is given by $\int_{-\infty}^{\infty} tq(t)\,dt$. Using our answer to part (a):

$$\int_{-\infty}^{\infty} tq(t)\,dt = \int_{-\infty}^{0} 0\,dt + \int_0^2 \frac{c}{2}t^2\,dt + \int_2^{\infty} c\,te^{2-t}\,dt = \int_0^2 \frac{c}{2}t^2\,dt + \int_2^{\infty} c\,te^{2-t}\,dt.$$

Answer: $\int_0^2 \frac{c}{2}t^2\,dt + \int_2^{\infty} c\,te^{2-t}\,dt$

c. [2 points] Use the fact that $Q(t)$ is a cumulative distribution function to find the value of c.

Solution: Since $Q(t)$ is a cumulative distribution function, we must have $\lim_{t\to\infty} Q(t) = 1$. Using the formula for $Q(t)$ for $t > 2$,

$$\lim_{t\to\infty} Q(t) = \lim_{t\to\infty} (2c - c\,e^{2-t}) = 2c.$$

Therefore $2c = 1$, so that $c = \frac{1}{2}$.

Answer: $c = \frac{1}{2}$

d. [2 points] Circle the **one** correct answer below that completes the following sentence:

"The quantity $Q(5)$ represents...

(i) *...the probability that it takes exactly 5 days for Liban to write a song."*

(ii) *...the probability that it takes more than 5 days for Liban to write a song."*

[✓ (iii)] *...the probability that it takes 5 days or less for Liban to write a song."*

(iv) *...the approximate probability that it takes between 4.5 and 5.5 days for Liban to write a song."*

(v) NONE OF THESE

Solution: We know that $Q(5)$ and $q(5)$ are related by the formula $Q(5) = \int_{-\infty}^{5} q(t)\,dt$, and this integral represents the probability that it takes 5 days or less for Liban to write a song. To explain why the other choices are incorrect:

In general, the probability that it takes Liban between a and b days to write a song is the quantity

$$Q(b) - Q(a) = \int_a^b q(t)\,dt.$$

Thus the probability that it takes exactly 5 days for Liban to write a song must be

$$Q(5) - Q(5) = \int_5^5 q(t)\,dt = 0.$$

But $Q(5) \neq 0$ since we know $Q(t)$ is a nondecreasing function (as it is a cdf), and so we have $Q(5) \geq Q(2) = 0.5 > 0$, using our value of c from part (c). So (i) is incorrect.

The probability that it takes more than 5 days for Liban to write a song is $1 - Q(5) = \int_5^{\infty} q(t)\,dt$. We have $Q(2) = 0.5$. The formula for $Q(t)$ shows that it is strictly increasing, so $Q(5) > 0.5$, and thus $1 - Q(5) < 0.5$. This means $Q(5) \neq 1 - Q(5)$, so (ii) is incorrect.

The approximate probability that it takes between 4.5 and 5.5 days for Liban to write a song is a standard interpretation of the quantity $q(5)$, which describes a pdf, not a cdf. So (iv) is also incorrect.

4. [8 points] Suppose a_n and b_n are sequences of positive numbers, defined for $n = 1, 2, 3\ldots$, satisfying the following conditions:

- $a_n \leq \frac{1}{n^{1/2}}$
- $b_n \geq \frac{1}{n}$

For each statement below, circle ALWAYS if the statement is always true, SOMETIMES if the statement can be true or false depending on the specifics of a_n or b_n, and NEVER if the statement is false for all specific a_n or b_n.

a. [1 point] $\lim\limits_{n\to\infty} a_n = 0$.

[✓ ALWAYS] SOMETIMES NEVER

b. [1 point] $\lim\limits_{n\to\infty} b_n = 0$.

ALWAYS [✓ SOMETIMES] NEVER

c. [1 point] a_n is bounded.

[✓ ALWAYS] SOMETIMES NEVER

d. [1 point] b_n is monotone.

ALWAYS [✓ SOMETIMES] NEVER

e. [1 point] $\sum\limits_{n=1}^{\infty} a_n$ converges.

ALWAYS [✓ SOMETIMES] NEVER

f. [1 point] $\sum\limits_{n=1}^{\infty} b_n$ converges.

ALWAYS SOMETIMES [✓ NEVER]

g. [1 point] $\sum\limits_{n=1}^{\infty} (-1)^n b_n$ converges.

ALWAYS [✓ SOMETIMES] NEVER

h. [1 point] $\sum\limits_{n=1}^{\infty} (-1)^n a_n$ converges.

ALWAYS [✓ SOMETIMES] NEVER

9. [12 points] Katydyd is on vacation from her strenuous bakery job, and is at the beach. She is building a tower out of sand, but periodically sand falls off the top of the tower. Each time sand falls off the tower it gets 25% shorter, and between times sand falls off the top of the tower Katydyd increases its height by 2 inches.

a. [5 points] Let M_n denote the height of Katydyd's tower, in inches, immediately *before* the n^{th} time sand falls off the top of it. Before the first time sand falls off the tower it has a height of 6 inches (so $M_1 = 6$). Find expressions for the values of M_2, M_3 and M_4. You do not need to simplify your expressions.

Solution:

$$\begin{aligned}M_1 &= 6\\ M_2 &= 0.75M_1 + 2 = 0.75(6) + 2\\ M_3 &= 0.75M_2 + 2 = 0.75^2(6) + 0.75(2) + 2\\ M_4 &= 0.75M_4 + 3 = 0.75^3(6) + 0.75^2(2) + 0.75(2) + 2\end{aligned}$$

Answer: $M_2 =$ $0.75(6) + 2$

Answer: $M_3 =$ $0.75^2(6) + 0.75(2) + 2$

Answer: $M_4 =$ $0.75^3(6) + 0.75^2(2) + 0.75(2) + 2$

b. [5 points] Find a closed-form expression for M_n. Closed form means your answer should not include ellipses or sigma notation, and should NOT be recursive. You do not need to simplify your expression.

Solution:

$$\begin{aligned}M_n &= 0.75^{n-1}(6) + 2(0.75^{n-2} + \cdots + 0.75 + 1)\\ &= 0.75^{n-1}(6) + \frac{2(1 - 0.75^{n-1})}{1 - 0.75}\end{aligned}$$

Answer: $M_n =$ $0.75^{n-1}(6) + \frac{2(1 - 0.75^{n-1})}{1 - 0.75}$

c. [2 points] If Katydyd were to keep doing this indefinitely, what height would her tower approach, in inches, in the long run?

Solution:

$$\lim_{n\to\infty} M_n = \lim_{n\to\infty} 0.75^{n-1}(6) + \frac{2(1 - 0.75^{n-1})}{1 - 0.75} = \frac{2}{0.25} = 8$$

Answer: **8**

11. [12 points] For the following questions, determine if the statement is ALWAYS true, SOMETIMES true, or NEVER true, and circle the corresponding answer. Justification is not required.

a. [2 points] Let $F(x)$ be a cumulative distribution function (cdf) that is continuous for all x. Then, the series $\sum_{n=1}^{\infty}(-1)^n(1-F(n))$ converges.

Circle one: [✓ **ALWAYS**] **SOMETIMES** **NEVER**

b. [2 points] Let $f(x)$ be a non-negative, continuous function for all $x \geq 1$, and suppose that $f(x)$ is decreasing. Then, the integral $\int_1^{\infty} \frac{f(x)}{x}\,\mathrm{d}x$ converges.

Circle one: **ALWAYS** [✓ **SOMETIMES**] **NEVER**

c. [2 points] Let $g(x)$ be a continuous function and suppose that for all n, $s_n = g(n)$. If $\int_1^{\infty} g(x)\,\mathrm{d}x$ diverges, then $\sum_{n=1}^{\infty} s_n$ also diverges.

Circle one: **ALWAYS** [✓ **SOMETIMES**] **NEVER**

d. [2 points] Let $h(x)$ be a non-negative, continuous function for all x, and suppose that $h(x)$ is decreasing. Let $a_n = h(n)$ for all n. If $\int_1^{\infty} xh(x)\,\mathrm{d}x$ converges, then the series $\sum_{n=1}^{\infty} a_n$ also converges.

Circle one: [✓ **ALWAYS**] **SOMETIMES** **NEVER**

e. [2 points] Let $b_n \geq 0$ and $c_n \geq 0$ for all n. Suppose that the series $\sum_{n=1}^{\infty} b_n$ converges and that the sequence c_n also converges. Then, the series $\sum_{n=1}^{\infty} b_n 2^{c_n}$ diverges.

Circle one: **ALWAYS** **SOMETIMES** [✓ **NEVER**]

f. [2 points] Suppose that d_n is a monotonic decreasing sequence of positive numbers that converges to 0. Furthermore, assume that $\lim_{n\to\infty} \frac{d_n}{1/n^2} = 5$. Then, the series $\sum_{n=1}^{\infty}(-1)^n d_n$ is conditionally convergent.

Circle one: **ALWAYS** **SOMETIMES** [✓ **NEVER**]

6. [14 points] Determine whether each of the following series converge conditionally, converges absolutely, or diverges and circle the appropriate answer. **Fully justify** your answer including using **proper notation** and showing mechanics of any tests you use.

a. [7 points]

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{6+\sqrt{n}}$$

Circle one: **Absolutely Converges** [✓ **Conditionally Converges**] **Diverges**

Solution: First, we use the alternating series test to show the series itself converges: Let $a_n = \frac{1}{6+\sqrt{n}}$. It is easily verifiable that

$$\lim_{n\to\infty} a_n = 0,$$
$$a_n > 0 \text{ for } n \geq 1$$
$$a_n > a_{n+1} > 0.$$

So, by the Alternating Series Test, the series $\sum_{n=1}^{\infty}(-1)^n a_n$ converges.

Now, let's show that $\sum_{n=1}^{\infty} a_n$ diverges: Note that for $n \geq 1$,

$$\frac{1}{6+\sqrt{n}} \geq \frac{1}{7\sqrt{n}}.$$

By the p-test for series($p = \frac{1}{2}$), $\sum_{n=1}^{\infty} \frac{1}{7\sqrt{n}}$ diverges. So, by the (Direct) Comparison Test $\sum_{n=1}^{\infty} a_n$ diverges.

6. (continued) Here is a reproduction of the instructions for the problem:
Determine whether each of the following series converge conditionally, converges absolutely, or diverges and circle the appropriate answer. **Fully justify** your answer including using **proper notation** and showing mechanics of any tests you use.

b. [7 points]

$$\sum_{n=1}^{\infty} \frac{n^2 + 50n \sin 2n}{n^{7/2}}$$

Circle one: [✓ **Absolutely Converges**] **Conditionally Converges** **Diverges**

Solution: Set $a_n = \dfrac{n^2 + 50n \sin 2n}{n^{7/2}}$. Note that a_n can be positive or negative, but does not alternate. We have for $n \geq 1$,

$$|a_n| \leq \frac{n^2 + 50n}{n^{7/2}} \leq \frac{51n^2}{n^{7/2}} = 51n^{-3/2}.$$

By the p-test for series ($p = 3/2$, $\sum_{n=1}^{\infty} \frac{51}{n^{3/2}}$ converges. So, by the (Direct) Comparison Test, $\displaystyle\sum_{n=1}^{\infty} |a_n|$ converges. So $\displaystyle\sum_{n=1}^{\infty} a_n$ is absolutely convergent.