Math 115 — Practice for Exam 2

Generated November 3, 2024

UMID: **SOLUTIONS** ______________________ Initials: ______

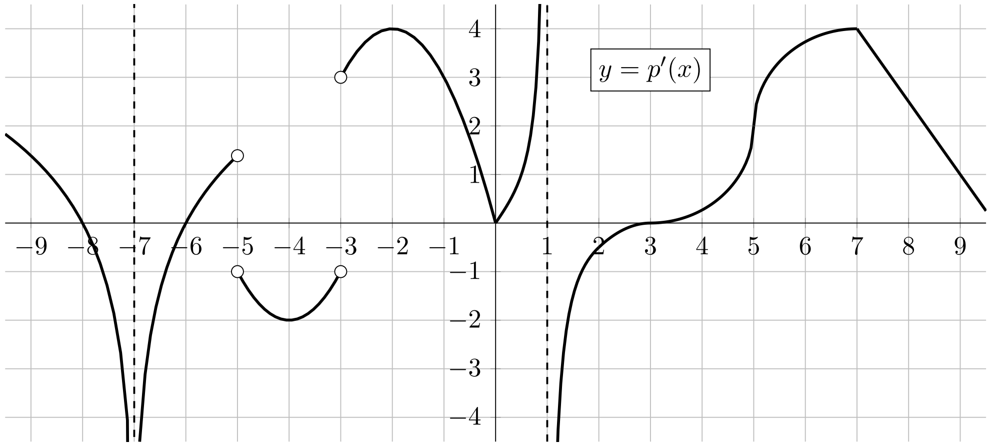
Instructor: ______________________ Section Number: ______

1. This exam has 9 questions. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
2. Please read the instructions for each individual exercise carefully. One of the skills being tested on this exam is your ability to interpret questions, so instructors will not answer questions about exam problems during the exam.
3. Show an appropriate amount of work (including appropriate explanation) for each exercise so that the graders can see not only the answer but also how you obtained it. Include units in your answers where appropriate.
4. You may not use a calculator. You are allowed one double-sided 8×11 inch page of handwritten notes.
5. If you use graphs or tables to obtain an answer, be certain to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
6. You must use the methods learned in this course to solve all problems.
7. You are responsible for reading and following all directions provided for each exam in this course.

Semester	Exam	Problem	Name	Points	Score
Fall 2022	3	9		12	
Fall 2021	2	3	toy airplane	15	
Fall 2018	2	3		12	
Fall 2022	2	9	bowtie	7	
Winter 2024	2	4	Stan Lee 2	8	
Winter 2024	2	5	Stan Lee 3	4	
Fall 2004	2	9	merry go round	12	
Winter 2019	2	7		8	
Winter 2008	2	5	microphone	16	
Total				94	

Recommended time (based on points): 94 minutes

9. [12 points] A function $p(x)$ is continuous on $(-\infty, \infty)$. Part of its **derivative** $p'(x)$ is shown below.



a. [2 points] At which of the following x-value(s) does $p(x)$ have a critical point?

[-8] [-3] -2 [1] 7

b. [2 points] At which of the following x-value(s) does $p(x)$ have a local minimum?

-8 -4 [-3] 0 [3]

c. [2 points] On which of the following interval(s) is $p(x)$ increasing on the entire interval?

$(-4, -3)$ [$(-3, 0)$] $(1, 2)$ [$(3, 7)$]

d. [2 points] On which of the following interval(s) is $p(x)$ concave up on the entire interval?

$(-5, -3)$ [$(0, 1)$] [$(3, 7)$] $(7, 9)$

e. [2 points] On which of the following interval(s) is the product $p'(x) \cdot p''(x)$ negative on the entire interval?

$(-5, -4)$ [$(-2, 0)$] [$(1, 3)$] $(3, 5)$

f. [2 points] At which of the following x-value(s) does $p(x)$ have both a local extremum and an inflection point?

-7 [-5] -3 0 3 NONE OF THESE

3. [15 points] Let the differentiable function $h(t)$ represent the height in inches (in) of a toy airplane above the ground at time t seconds (sec). Below is a table of some values for $h(t)$ and $h'(t)$. Assume that $h(t)$ is invertible, and that $h'(t)$ is differentiable for $t > 0$.

t	0	2	4	6	8
$h(t)$	28	19	11	8	4
$h'(t)$	-5	-4	-2	-1.5	-0.5

For parts **a.**–**d.**, you do not need to show work, but partial credit can be earned from work shown. You do not need to simplify numerical answers.

a. [3 points] Approximate $h''(8)$. Include units.

Solution:

$$h''(8) \approx \frac{h'(8) - h'(6)}{8-6} = \frac{-0.5 - (-1.5)}{2} = \frac{1}{2}.$$

Answer: 0.5 **in/sec^2**

b. [3 points] Find a formula for the linear approximation $L(t)$ to the function $h(t)$ at $t = 2$.

Solution: $L(t) = h(2) + h'(2)(t-2) = 19 - 4(t-2) = -4t + 27$

Answer: $L(t) =$ $19 - 4(t-2) = -4t + 27$

c. [2 points] Use your answer from the previous part to approximate $h(1.9)$. Include units.

Solution: $h(1.9) \approx L(1.9) = 19 - 4(1.9-2) = 19 - 4(-0.1) = 19.4$

Answer: 19.4 **inches**

d. [2 points] Compute the **exact** value of $(h^{-1})'(8)$. (You do not need to include units.)

Solution: $(h^{-1})'(8) = \dfrac{1}{h'(h^{-1}(8))} = \dfrac{1}{h'(6)} = \dfrac{1}{-1.5}.$

Answer: $\dfrac{1}{-1.5} = -\dfrac{2}{3}$

e. [3 points] Suppose that $(h^{-1})'(3) = -9$. Complete the following sentence to give a practical interpretation of this equation.

When the toy airplane is at a height of 3 inches, to descend an additional 0.1 inches . . .

Solution: ... will take about 0.9 seconds.

f. [2 points] Note that $h(t)$ satisfies the hypotheses of the Mean Value Theorem on $[0, 8]$. Complete the following sentence about what the conclusion of this theorem implies is true.

At some time between $t = 0$ and $t = 8$, the height of the toy airplane is . . .

(circle one) INCREASING **[DECREASING]** at a rate of **3** in/sec.

3. [12 points] Assume the function $h(t)$ is invertible and $h'(t)$ is differentiable. Some of the values of the function $y = h(t)$ and its derivatives are shown in the table below

t	0	1	2	3	4
$h(t)$	-2	2	3	4	8
$h'(t)$	3.5	0.5	2.5	1.5	5
$h''(t)$	6	0.25	0.3	-0.4	0.6

Use the values in the table to compute the exact value of the following mathematical expressions. If there is not enough information provided to find the value, write NI. If the value does not exist, write DNE. **Show all your work**.

a. [3 points] Let $a(t) = h(t^2 - 5)$. Find $a'(3)$.

Solution: Since $a'(t) = 2th'(t^2 - 5)$, then $a'(3) = 6h'(4) = 6(5) = 30$

Answer: 30

b. [3 points] Let $b(t) = \frac{h(t)}{t^2}$. Find $b'(4)$.

Solution: Since

$$b'(t) = \frac{h'(t)t^2 - 2th(t)}{t^4} \quad \text{then} \quad b'(4) = \frac{16h'(4) - 8h(4)}{256} = \frac{16(5) - 8(8)}{256} = \frac{16}{256} = \frac{1}{16}.$$

Answer: $\frac{1}{16}$

c. [3 points] Let $c(y) = h^{-1}(y)$. Find $c'(2)$.

Solution: Since $c'(y) = \frac{1}{h'(h^{-1}(y))}$ then $c'(2) = \frac{1}{h'(h^{-1}(2))} = \frac{1}{h'(1)} = 2.$

Answer: 2

d. [3 points] Let $g(t) = \ln(1 + 2h'(t))$. Find $g'(0)$.

Solution: Since $g'(t) = \frac{2h''(t)}{1 + 2h'(t)}$ then $g'(0) = \frac{2h''(0)}{1 + 2h'(0)} = \frac{2(6)}{1 + 2(3.5)} = \frac{12}{8} = 1.5$

Answer: 1.5

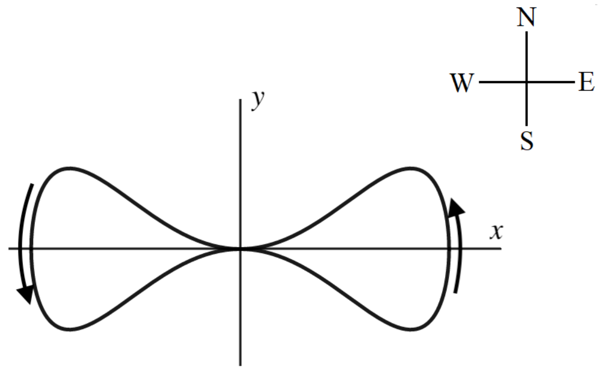
9. [7 points]

You are on a hiking trip, following the path modeled by the curve $\mathcal{B}$ defined by the equation

$$y^2 = x^4(1-x^2).$$

Note that

$$\frac{dy}{dx} = \frac{x^3(2-3x^2)}{y}.$$



The graph of $\mathcal{B}$ is shown to the right. You begin your hike at $(0,0)$, then:

- travel East and around the loop on the right as shown by the arrow, returning to $(0,0)$, then
- travel West and around the loop on the left as shown by the arrow, returning to $(0,0)$.

a. [5 points] Using calculus, find the coordinates of all the other points (x,y) on your path (that is, other than $(0,0)$), where you travel directly East or directly West. Show your work.
Note that you can use the graph to determine how many points you are looking for.

Solution: We look for where the numerator of $\frac{dy}{dx}$ is 0, i.e. $x^3(2-3x^2)=0$. We ignore the solution $(0,0)$, so we need $2-3x^2=0$ or $x=\pm\sqrt{\frac{2}{3}}$. Then

$$y^2 = \left(\pm\sqrt{\frac{2}{3}}\right)^4\left(1-\left(\pm\sqrt{\frac{2}{3}}\right)^2\right)$$

$$y^2 = \frac{4}{9}\left(1-\frac{2}{3}\right) = \frac{4}{9}\cdot\frac{1}{3} = \frac{4}{27}$$

so $y = \pm\frac{2}{3\sqrt{3}}$. We can see which direction we are traveling at these points from the graph.

Answer: travel East at $\left(\sqrt{\frac{2}{3}}, -\frac{2}{3\sqrt{3}}\right), \left(-\sqrt{\frac{2}{3}}, -\frac{2}{3\sqrt{3}}\right)$

Answer: travel West at $\left(\sqrt{\frac{2}{3}}, \frac{2}{3\sqrt{3}}\right), \left(-\sqrt{\frac{2}{3}}, \frac{2}{3\sqrt{3}}\right)$

b. [2 points] Using calculus, find the coordinates of all the points (x,y) on your path where you travel directly North or directly South. Note that, as shown by the graph, $(0,0)$ is not one of these points. Show your work.

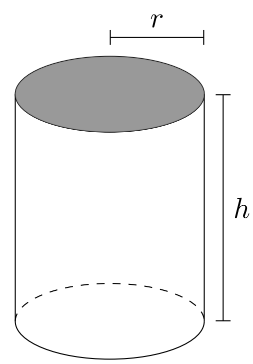
Solution: We look for where the denominator of $\frac{dy}{dx}$ is 0, i.e. $y=0$. Then $0=x^4(1-x^2)$ so either $x=0$ or $x=\pm1$. We ignore $(0,0)$, so the points are $(1,0)$ and $(-1,0)$.

Answer: travel North at $(1,0)$

Answer: travel South at $(-1,0)$

4. [8 points]

Alana is designing a new prototype for her Stan Lee cups. The new cups are cylindrical in shape, with metal sides and base, and a circular lid made from silicone. If the cylinder has height h centimeters, and radius r centimeters, then the surface area of the metal part is $2\pi rh + \pi r^2$ square centimeters, and the surface area of the silicone part is πr^2. The metal costs 2 cents per square centimeter, and the silicone costs 3 cents per square centimeter. Alana spends a total of 300 cents on materials for each cup.



a. [3 points] Find a formula for h in terms of r.

Solution: The total cost in cents of materials for each cup is

$$300 = 2\left(2\pi rh + \pi r^2\right) + 3\pi r^2 = 4\pi rh + 5\pi r^2.$$

Therefore, $4\pi rh = 300 - 5\pi r^2$, so $h = \dfrac{300 - 5\pi r^2}{4\pi r} = \dfrac{75}{\pi r} - \dfrac{5r}{4}$.

Answer: $h = \dfrac{300 - 5\pi r^2}{4\pi r}$

b. [1 point] Recall that the volume of a cylinder of radius r and height h is $V = \pi r^2 h$. Write a formula for $V(r)$, the volume of one of the cups in cubic centimeters, as a function of r only. *Your formula should not include the letter h.*

Answer: $V(r) = \pi r^2 \cdot \dfrac{300 - 5\pi r^2}{4\pi r} = r \cdot \dfrac{300 - 5\pi r^2}{4}$

c. [4 points] Alana wants to ensure that the height of a cup is at most 2 and a half times its radius, that is, she wants $h \leq 2.5r$. Given this constraint, find the domain of $V(r)$ in the context of this problem.

Solution: In the context of this problem, the height should be positive, so we have the constraints $0 < h \leq 2.5r$. Substituting in our expression for h from part **a.**, we get:

$$0 < \frac{300 - 5\pi r^2}{4\pi r} \leq 2.5r$$
$$0 < 300 - 5\pi r^2 \leq 10\pi r^2$$
$$5\pi r^2 < 300 \leq 15\pi r^2.$$

Therefore, we must have

$$\frac{300}{15\pi} \leq r^2 < \frac{300}{5\pi}, \qquad \text{which means} \qquad \sqrt{\frac{20}{\pi}} \leq r < \sqrt{\frac{60}{\pi}}.$$

Answer: $\left[\sqrt{\dfrac{20}{\pi}}, \sqrt{\dfrac{60}{\pi}}\right)$

5. [4 points] One of Alana's interns suggests that instead of trying to fix the cost of their Stan Lee cups and maximize the volume, they should instead fix the volume and try to minimize the cost. Assuming they try to match their competitor's standard cup size of 500ml, their cost of producing each cup is

$$C(r) = \frac{2000}{r} + 5\pi r^2 \quad \text{dollars},$$

where r is the radius of the cup. Assuming the only constraint on r is that $r > 0$, what is the cup radius that minimizes their cost? *Show all your work.*

Solution: We need to minimize $C(r)$ over the domain $(0, \infty)$. Taking a derivative, we find

$$C'(r) = -2000r^{-2} + 10\pi r,$$

so $C(r)$ is differentiable everywhere on $(0, \infty)$. Setting $C'(r) = 0$ and solving we get:

$$\frac{2000}{r^2} = 10\pi r,$$
$$2000 = 10\pi r^3,$$
$$\frac{200}{\pi} = r^3,$$

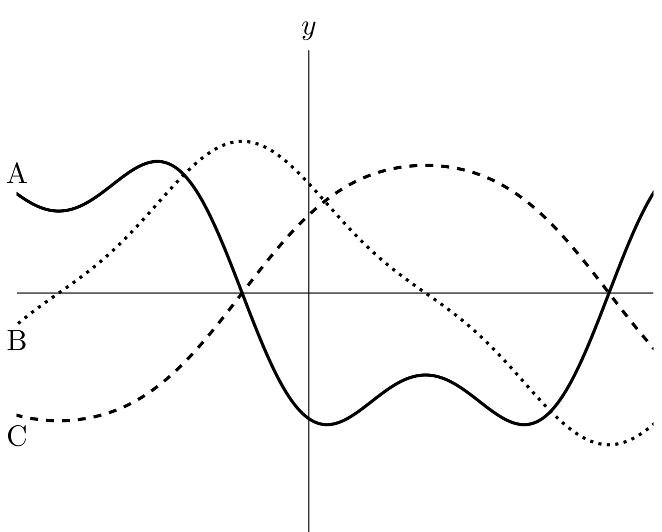
so the only critical point of $C(r)$ on $(0, \infty)$ is $r = \sqrt[3]{\frac{200}{\pi}} = \left(\frac{200}{\pi}\right)^{1/3}$. Since

$$\lim_{x\to 0^+} C(r) = \infty = \lim_{x\to\infty} C(r),$$

this critical point must indeed be the global minimum of $C(r)$ on $(0, \infty)$.

Answer: $r =$ $\left(\frac{200}{\pi}\right)^{1/3}$ centimeters

6. [4 points] Shown below are portions of the graphs of the functions $y = f(x)$, $y = f'(x)$, and $y = f''(x)$. Determine which graph is which, and then, on the answer lines below, indicate after each function the letter A, B, or C that corresponds to its graph. No work or justification is needed.

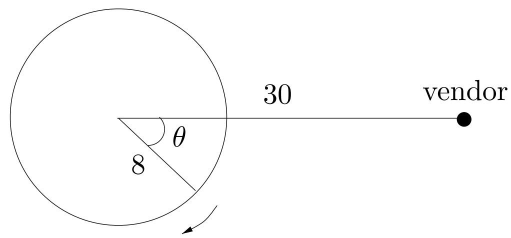


Answer: $f(x)$: C

$f'(x)$: B

$f''(x)$: A

9. (2+4+6 points) You have been searching for the cotton candy vendor all day at the carnival. As you board the merry-go-round, you spot the candy man. Unfortunately, you are stuck on the merry-go-round. The vendor's stand is 30 feet from the center of the merry-go-round, and you begin your ride directly on the line of sight between the center of the merry-go-round and the vendor. The merry-go-round has a radius of 8 feet and is turning at a rate of $\frac{\pi}{60}$ radians/second.



(a) How long does it take for the merry-go-round to rotate $\frac{\pi}{6}$ radians?

$t = 10$ seconds.

(b) How far are you from the vendor when the merry-go-round has rotated $\frac{\pi}{6}$ radians? [The law of cosines may help here. It states that given a triangle of side lengths a, b, and c with angle θ between sides a and b, then one has $c^2 = a^2 + b^2 - 2ab\cos\theta$.]

Use the law of cosines with $a = 8$, $b = 30$, $\theta = \frac{\pi}{6}$, and c the distance between you and the vendor. So $c = 23.42$ feet.

(c) How fast is the distance between you and the vendor changing when the merry-go-round has rotated $\frac{\pi}{6}$ radians?

Take the derivative of the law of cosines with respect to t:

$$2c\frac{dc}{dt} = 2ab\sin(\theta)\frac{d\theta}{dt}.$$

Solving this equation for $\frac{dc}{dt}$ and using that $a = 8$, $b = 30$, $c = 23.42$, $\theta = \frac{\pi}{6}$, and $\frac{d\theta}{dt} = \frac{\pi}{60}$ we obtain that $\frac{dc}{dt} = 0.27$ feet/second.

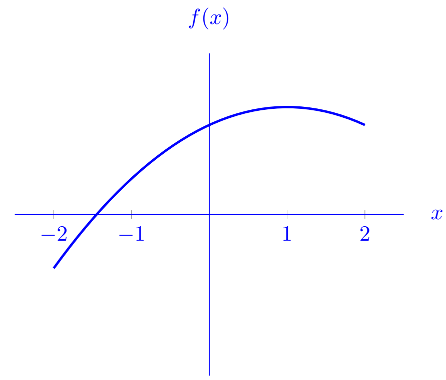
7. [8 points] For each part, draw a function on the given axes that satisfies the given conditions. Or, if no such function exists, write DNE and provide a brief explanation.

Make sure your graphs are unambiguous and that the domain of each graph is clear.

a. [2 points]

A differentiable function $f(x)$ with domain $[-2, 2]$ that has a global maximum at $x = 1$ and $f''(x) \leq 0$.

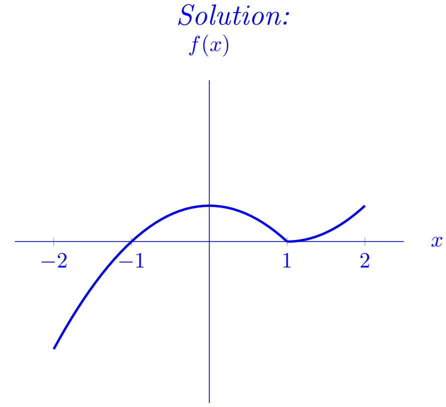
Solution:



b. [3 points]

A continuous function $f(x)$ with domain $[-2, 2]$ that has both a local minimum at $x = 1$ and an inflection point at $x = 1$.

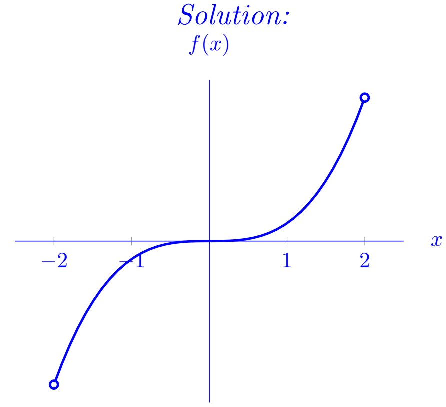
Solution:



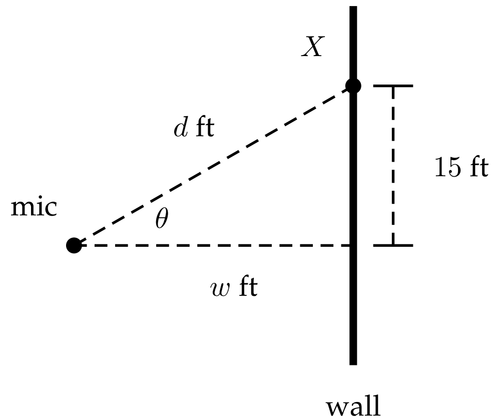
c. [3 points]

A continuous function $f(x)$ with domain $(-2, 2)$ that has exactly one critical point and no global extrema. Note that this domain differs from those in previous parts.

Solution:



5. (16 points) A directional microphone is mounted on a stand facing a wall. The sensitivity S of the microphone to sounds at point X on the wall is inversely proportional to the square of the distance d from the point X to the mic, and directly proportional to the cosine of the angle θ. That is, $S = K\frac{\cos\theta}{d^2}$ for some constant K. (See the diagram below.) How far from the wall should the mic be placed to maximize sensitivity to sounds at X?



We are given that $S = K\frac{\cos\theta}{d^2}$. Also, from the definition of cosine we see $\cos\theta = \frac{w}{d}$. By the Pythagorean theorem, $d^2 = w^2 + 15^2 = w^2 + 225$. Therefore

$$S = K\frac{\cos\theta}{d^2} = K\frac{\frac{w}{d}}{w^2+225} = K\frac{w}{(w^2+225)^{3/2}}.$$

Differentiating, we get

$$\frac{dS}{dw} = K\frac{(w^2+225)^{3/2} - 3w^2(w^2+225)^{1/2}}{(w^2+225)^3}.$$

The derivative is defined for all w and is only equal to zero when the numerator is zero. Factoring the common factor of $(w^2+225)^{1/2}$ gives

$$(w^2+225)^{1/2}(w^2+225-3w^2),$$

and since (w^2+225) is never zero, we must have

$$2w^2 = 225,$$

or

$$w = \pm\sqrt{\frac{225}{2}} = \pm\frac{15}{\sqrt{2}}.$$

Since w is a length, we discard the negative root, and now must test the one critical point $w = \frac{15}{\sqrt{2}}$. Note that for $w < \frac{15}{\sqrt{2}}$, the first derivative is positive, and for $w > \frac{15}{\sqrt{2}}$ the derivative is negative. Thus, by the first derivative test, $w = \frac{15}{\sqrt{2}}$ is a local maximum. Since the function is continuous and this is the only critical point on the domain, $w = \frac{15}{\sqrt{2}}$ is the global maximum. The mic should be placed $\frac{15}{\sqrt{2}} \approx 10.6$ feet from the wall to maximize the sensitivity to sounds at X.