Math 116 — Practice for Exam 1

Generated February 3, 2025

Name: SOLUTIONS

Instructor: ______________________ Section Number: ________

1. This exam has 9 questions. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
2. Do not separate the pages of the exam. If any pages do become separated, write your name on them and point them out to your instructor when you hand in the exam.
3. Please read the instructions for each individual exercise carefully. One of the skills being tested on this exam is your ability to interpret questions, so instructors will not answer questions about exam problems during the exam.
4. Show an appropriate amount of work (including appropriate explanation) for each exercise so that the graders can see not only the answer but also how you obtained it. Include units in your answers where appropriate.
5. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
6. If you use graphs or tables to obtain an answer, be certain to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
7. You must use the methods learned in this course to solve all problems.

Semester	Exam	Problem	Name	Points	Score
Fall 2021	1	3	race	13	
Fall 2011	1	7	AC current	13	
Winter 2024	1	1		16	
Winter 2021	1	7		6	
Winter 2021	1	8		13	
Fall 2022	1	2		15	
Winter 2018	1	8		14	
Winter 2015	1	2	dog bowl	13	
Winter 2023	1	8	asteroid	12	
Total				115	

Recommended time (based on points): 104 minutes

3. [13 points] Miley and Kylie see their friend Brian in the distance and decide to race to see who can reach him first. However, they see Brian begin pacing back and forth so depending on when they start the race, they will run a different amount. The distance they run, in meters, if the race starts t seconds after Brian begins pacing is

$$L(t) = 25 + 4 \int_{-\left(\frac{\pi}{2}\right)^{\frac{1}{3}}}^{t^3} \cos(r^3) r^2 dr$$

Throughout this problem, please give answers in **exact form** and include **units.**

a. [4 points] If Miley and Kylie start the race immediately as Brian begins pacing, what distance will they run? Evaluate any integrals in your answer and remember to include units.

Solution: The initial value is $L(0)$. Substituting $w = r^3$,

$$\begin{aligned} L(0) &= 25 + 4 \int_{-\left(\frac{\pi}{2}\right)^{\frac{1}{3}}}^{0} \cos(r^3) r^2 dr \\ &= 25 + \frac{4}{3} \int_{-\frac{\pi}{2}}^{0} \cos w dw = 25 + \frac{4}{3}\left(\sin w \Big|_{-\frac{\pi}{2}}^{0} \right) \\ &= 25 + \frac{4}{3}(0 - (-1)) = 25 + \frac{4}{3} \text{ meters.} \end{aligned}$$

b. [6 points] Miley and Kylie decide they will start the race at the smallest strictly positive time t (i.e. smallest t with $t > 0$) such that $L'(t) = 0$. Find the time at which they will start the race. Make sure to include units.

Solution: Taking the derivative using the chain rule,

$$L'(t) = 4\cos((t^3)^3)(t^3)^2 3t^2 = 12t^8 \cos(t^9).$$

Since $t^8 \geq 0$ for $t \geq 0$, the smallest strictly positive time for which $L'(t) = 0$ occurs at the smallest positive time for which the cosine term is 0. This happens when $t^9 = \frac{\pi}{2}$. Solving this gives that the race will start at $t = \left(\frac{\pi}{2}\right)^{\frac{1}{9}}$ seconds.

c. [3 points] Miley and Kylie want to be able to compute $L(t)$ quickly, so they would like $L(t)$ rewritten in the form below. Write $L(t)$ in the form given below with appropriate expressions in place of the blanks.

$$L(t) = \underline{\qquad 25 + \tfrac{4}{3} \qquad} + \int_0^t \underline{\qquad 12r^8 \cos(r^9) \qquad}\, dr$$

7. [13 points] Household electricity in the United States is supplied in the form of an alternating current that varies sinusoidally with a frequency of 60 cycles per second (Hz). The voltage is given by the equation

$$E(t) = 170\sin(120\pi t),$$

where t is given in seconds and E is in volts.

a. [7 points] Using integration by parts, find $\int \sin^2\theta d\theta$. Show all work to receive full credit. (Hint: $\sin^2\theta + \cos^2\theta = 1$.)

Solution: We first note that $\int \sin^2\theta d\theta = \int \sin\theta(\sin\theta)d\theta$, so that we may take $u = \sin\theta, dv = \sin\theta d\theta$ (and $du = \cos\theta d\theta, v = -\cos\theta$). Then integration by parts gives

$$\begin{aligned}\int \sin^2\theta d\theta &= \sin\theta(-\cos\theta) - \int -\cos\theta(\cos\theta)d\theta \\ &= -\sin\theta\cos\theta + \int \cos^2\theta d\theta.\end{aligned}$$

Using the trig. identity given in the hint, we obtain

$$\int \sin^2\theta d\theta = -\sin\theta\cos\theta + \int \left(1-\sin^2\theta\right)d\theta = -\sin\theta\cos\theta + \int d\theta - \int \sin^2\theta.$$

The integral on the far right also appears on the left, so combining like terms, we get

$$\begin{aligned}2\int \sin^2\theta d\theta &= -\sin\theta\cos\theta + \int d\theta \\ \int \sin^2\theta d\theta &= \frac{1}{2}\left(-\sin\theta\cos\theta + \int d\theta\right) \\ &= -\frac{1}{2}\sin\theta\cos\theta + \frac{\theta}{2} + C.\end{aligned}$$

b. [6 points] Voltmeters read the root-mean-square (RMS) voltage, which is defined to be the square root of the average value of $[E(t)]^2$ over one cycle. Find the exact RMS voltage of household current.

Solution: Since the frequency of the current is 60 cycles per second, one cycle is completed every $\frac{1}{60}$ seconds. Thus

$$\begin{aligned}\text{RMS voltage} &= \sqrt{\frac{1}{\frac{1}{60}-0}\int_0^{\frac{1}{60}} E(t)^2 dt} \\ &= \sqrt{60\int_0^{\frac{1}{60}} 170^2 \sin^2(120\pi t)\, dt}.\end{aligned}$$

Substituting $w = 120\pi t, dw = 120\pi dt$, we get

$$\begin{aligned}\text{RMS voltage} &= \sqrt{60\int_{w(0)}^{w(\frac{1}{60})} 170^2 \sin^2(w)\cdot\frac{1}{120\pi}dw} \\ &= \sqrt{\frac{170^2}{2\pi}\int_0^{2\pi} \sin^2(w)\, dw}.\end{aligned}$$

Using the antiderivative we found in part (a) with $C = 0$, the Fundamental Theorem of Calculus gives

$$\begin{aligned}\text{RMS voltage} &= \sqrt{\frac{170^2}{2\pi}\left(-\frac{1}{2}\sin\theta\cos\theta+\frac{\theta}{2}\right)\Big|_0^{2\pi}} \\ &= \sqrt{\frac{170^2}{2\pi}\left((-\frac{1}{2}\sin(2\pi)\cos(2\pi)+\frac{2\pi}{2})-(-\frac{1}{2}\sin(0)\cos(0)+\frac{0}{2})\right)} = \frac{170}{\sqrt{2}}\text{ Volts.}\end{aligned}$$

Note that due to the periodicity of the sine function, the average value over one cycle could also have been computed over $0 \le t \le 1$ (or any other number of periods).

1. [16 points] Let $f(x)$ be a function that is **even** and **twice differentiable.** Some values of $f(x)$ and $f'(x)$ are given in the table below:

x	0	1	2	3	4
$f(x)$	-3	2	-1	0	5
$f'(x)$	0	4	$\sqrt{2}$	1	e

Use the table above to compute the **exact value** of the following integrals. If there is not enough information to determine the exact value of an integral, write "NEI." You need to evaluate all integrals completely, and your answers should not involve the letter f, but you do not need to simplify your final answers. Show all your work.

a. [3 points] $\displaystyle\int_{-2}^{2} f'(x)\,dx$

Solution: There are two possible ways to arrive at the answer:

Solution 1 ($f(x)$ is even)**:** Since $f(x)$ is an even function, we have $f(2) = f(-2)$. So, by the First Fundamental Theorem of Calculus, $\displaystyle\int_{-2}^{2} f'(x)\,dx = f(2) - f(-2) = 0$.

Solution 2 ($f'(x)$ is odd)**:** Since $f(x)$ is an even function, then $f(x) = f(-x)$ for all x. Taking derivatives of both sides and using the chain rule, $f'(x) = -f'(-x)$ for all x, so $f'(x)$ is an odd function. Therefore $\displaystyle\int_{-2}^{2} f'(x)\,dx = 0$ by symmetry.

Answer: 0

b. [4 points] $\displaystyle\int_{1}^{e^2} \frac{f'(\ln(t))}{t}\,dt$

Solution: We use a substitution $u = \ln(t)$, so that $du = \frac{1}{t}\,dt$. It follows that

$$\int_{1}^{e^2} \frac{f'(\ln(t))}{t}\,dt = \int_{0}^{2} f'(u)\,du = f(2) - f(0) = -1 - (-3) = 2.$$

Answer: 2

c. [4 points] $\int_1^3 (2w+1)f'(w)\,dw$

Solution: We integrate by parts:

$$\begin{aligned}\int_1^3 (2w+1)f'(w)\,dw &= (2w+1)f(w)\Big|_1^3 - \int_1^3 2f(w)\,dw \\ &= 7f(3) - 3f(1) - 2\int_1^3 f(w)\,dw \\ &= 0 - 6 - 2\int_1^3 f(w)\,dw.\end{aligned}$$

However, no information is given on an antiderivative of f, so we cannot evaluate $\int_1^3 f(w)\,dw$. Therefore the answer is NEI.

Answer: **NEI**

d. [5 points] $\int_1^2 2x^3 f''(x^2)\,dx$

Solution: First use a substitution $u = x^2$, so that $du = 2x\,dx$, and so we have

$$\int_1^2 2x^3 f''(x^2)\,dx = \int_1^4 uf''(u)\,du.$$

Now integrate by parts:

$$\begin{aligned}\int_1^4 uf''(u)\,du &= uf'(u)\Big|_1^4 - \int_1^4 f'(u)\,du \\ &= 4f'(4) - f'(1) - (f(4) - f(1)) \\ &= 4e - 4 - (5-2) \\ &= 4e - 7.\end{aligned}$$

Answer: $4e - 7$

7. [6 points] Split the function $\dfrac{5x^2-7x}{(x-1)^2(x+1)}$ into partial fractions with two or more terms. Do not integrate these terms. Be sure to show all work to obtain your partial fractions.

Solution: Let
$$\frac{5x^2-7x}{(x-1)^2(x+1)}=\frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{C}{x+1}.$$

Solution 1: To solve for B: multiply both sides by $(x-1)^2$, then put $x=1$:

$$\frac{5x^2-7x}{x+1}=A(x-1)+B+C\frac{x-1}{x+1},$$

$$\frac{-2}{2}=0+B+0, B=-1.$$

To solve for C: multiply both sides by $x+1$, then put $x=-1$:

$$\frac{5x^2-7x}{(x-1)^2}=A\frac{x+1}{x-1}+B\frac{x+1}{(x-1)^2}+C,$$

$$\frac{12}{4}=0+0+C, C=3.$$

To solve for A, clear the denominator, and compare the coefficients.

$$5x^2-7x=A(x-1)(x+1)-(x+1)+3(x-1)^2=(A+3)x^2-7x+(2-A).$$

Say we look at the coefficients of x^2:

$$5=A+3, A=2.$$

Therefore,
$$\frac{5x^2-7x}{(x-1)^2(x+1)}=\frac{2}{x-1}-\frac{1}{(x-1)^2}+\frac{3}{x+1}.$$

Solution 2: Clear the denominator.

$$5x^2-7x=A(x-1)(x+1)+B(x+1)+C(x-1)^2=(A+C)x^2+(B-2C)x+(-A+B+C).$$

Compare the coefficients.

$$A+C=5, B-2C=-7, -A+B+C=0.$$

Solve the system of equations. An easy way is to write $A=5-C$, $B=2C-7$, and we then have $-(5-C)+(2C-7)+C=0$. We have that $A=2, B=-1, C=3$.

8. [13 points] Let $f(x)$ be a twice differentiable function with

- $f(0)=1$.

7. [6 points] Split the function $\dfrac{5x^2-7x}{(x-1)^2(x+1)}$ into partial fractions with two or more terms. Do not integrate these terms. Be sure to show all work to obtain your partial fractions.

Solution: Let

$$\frac{5x^2-7x}{(x-1)^2(x+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1}.$$

Solution 1: To solve for B: multiply both sides by $(x-1)^2$, then put $x=1$:

$$\frac{5x^2-7x}{x+1} = A(x-1) + B + C\frac{x-1}{x+1},$$

$$\frac{-2}{2} = 0 + B + 0, B = -1.$$

To solve for C: multiply both sides by $x+1$, then put $x=-1$:

$$\frac{5x^2-7x}{(x-1)^2} = A\frac{x+1}{x-1} + B\frac{x+1}{(x-1)^2} + C,$$

$$\frac{12}{4} = 0 + 0 + C, C = 3.$$

To solve for A, clear the denominator, and compare the coefficients.

$$5x^2-7x = A(x-1)(x+1) - (x+1) + 3(x-1)^2 = (A+3)x^2 - 7x + (2-A).$$

Say we look at the coefficients of x^2:

$$5 = A+3, A = 2.$$

Therefore,

$$\frac{5x^2-7x}{(x-1)^2(x+1)} = \frac{2}{x-1} - \frac{1}{(x-1)^2} + \frac{3}{x+1}.$$

Solution 2: Clear the denominator.

$$5x^2-7x = A(x-1)(x+1) + B(x+1) + C(x-1)^2 = (A+C)x^2 + (B-2C)x + (-A+B+C).$$

Compare the coefficients.

$$A+C = 5, B-2C = -7, -A+B+C = 0.$$

Solve the system of equations. An easy way is to write $A = 5-C$, $B = 2C-7$, and we then have $-(5-C)+(2C-7)+C = 0$. We have that $A = 2, B = -1, C = 3$.

8. [13 points] Let $f(x)$ be a twice differentiable function with

- $f(0) = 1$.

- $f(\ln 2) = \frac{5}{4}$.
- $f'(0) = e$.
- $f'(\ln 2) = 2$.

a. [3 points] Compute the average value of $f'(x)$ on $[0, \ln 2]$.

Solution: The average value is

$$\frac{1}{\ln 2 - 0}\int_0^{\ln 2} f'(x)\ dx = \frac{1}{\ln 2}(f(\ln 2) - f(0)) = \frac{1}{\ln 2}(\frac{5}{4} - 1).$$

b. [5 points] Compute the exact value of $\displaystyle\int_0^{\ln 2} x f''(x)\ dx$.

Solution:

$$\begin{aligned}\int_0^{\ln 2} x f''(x)\ dx &= \big(xf'(x)\big)_0^{\ln 2} - \int_0^{\ln 2} f'(x)\ dx \\ &= (\ln 2 f'(\ln 2) - 0 f'(0)) - (f(\ln 2) - f(0)) \\ &= 2\ln 2 - \frac{5}{4} + 1.\end{aligned}$$

c. [5 points] Compute the exact value of $\displaystyle\int_0^{\ln 2} \frac{f'(x)}{\sqrt{9 - (f(x))^2}}\ dx$.

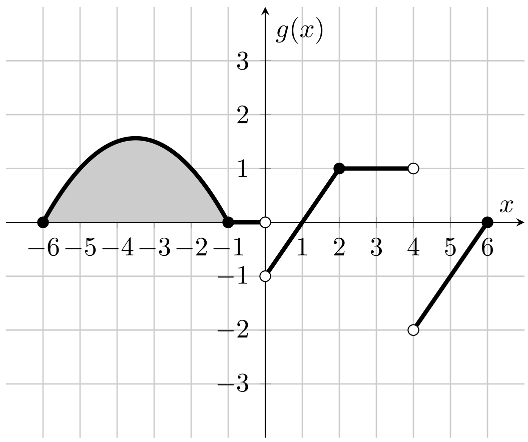
Solution: Let $w = f(x)$. The new upper and lower bounds are $f(0) = 1$ and $f(\ln 2) = \frac{5}{4}$ respectively.

$$\int_0^{\ln 2} \frac{f'(x)}{\sqrt{9 - (f(x))^2}}\ dx = \int_1^{\frac{5}{4}} \frac{1}{\sqrt{9 - w^2}}\ dw.$$

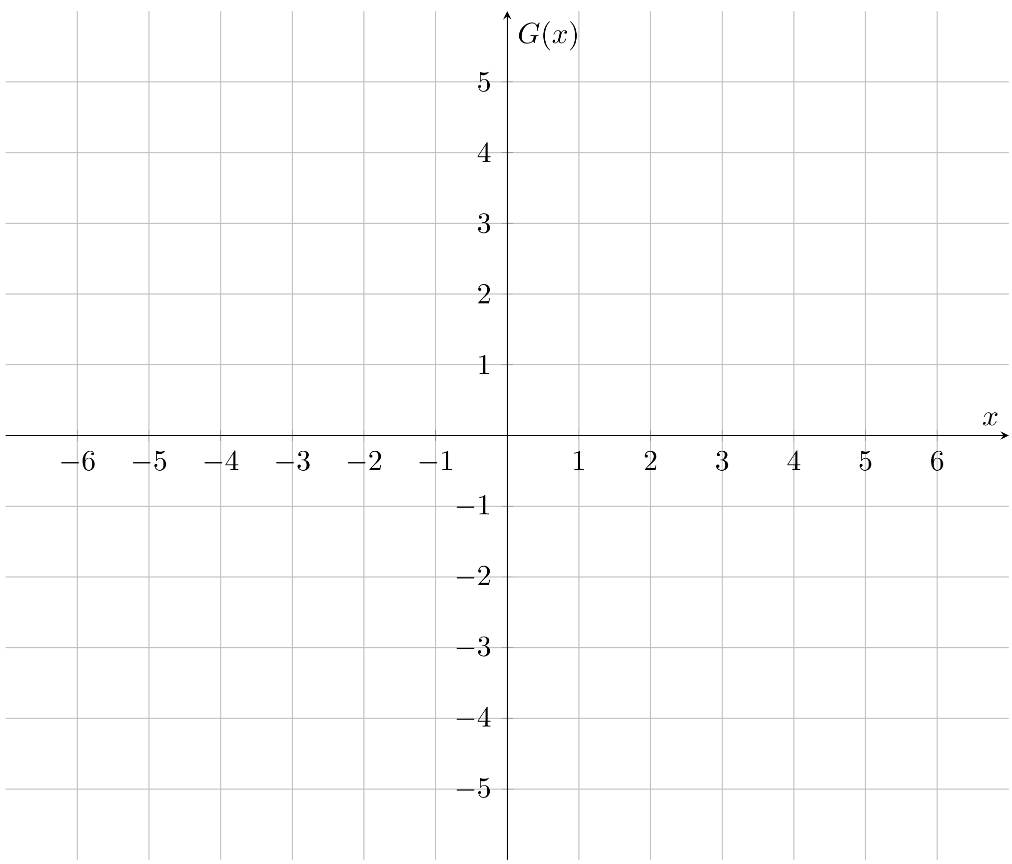
Let $w = 3\sin\theta$. The new θ-bounds are $\sin^{-1}\frac{1}{3}$ and $\sin^{-1}\frac{5}{12}$ respectively.

$$\begin{aligned}\int_1^{\frac{5}{4}} \frac{1}{\sqrt{9 - w}}\ dw &= \int_{\sin^{-1}\frac{1}{3}}^{\sin^{-1}\frac{5}{12}} \frac{3\cos\theta}{\sqrt{9 - 9\sin^2\theta}}\ d\theta \\ &= \int_{\sin^{-1}\frac{1}{3}}^{\sin^{-1}\frac{5}{12}} 1\ d\theta \\ &= \sin^{-1}\frac{5}{12} - \sin^{-1}\frac{1}{3}.\end{aligned}$$

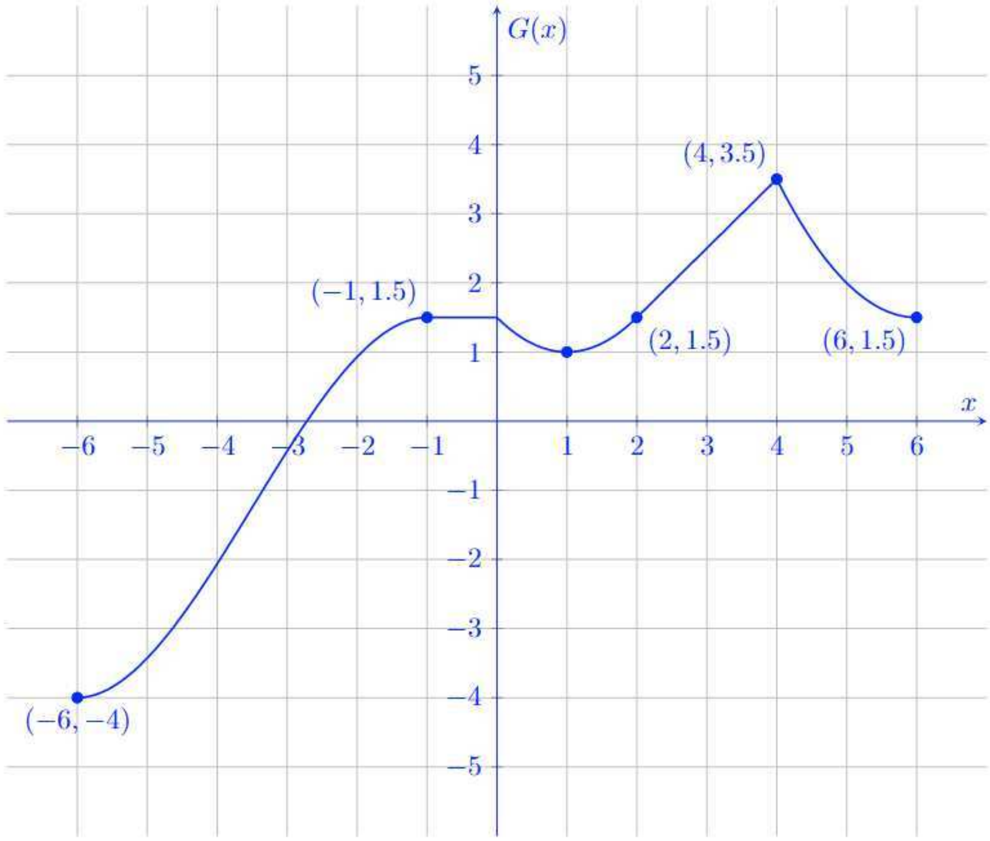
2. [15 points] The function $g(x)$ is graphed below. The area of the shaded region is 5.5. The function $g(x)$ is piecewise linear for $x > -1$.



On the axes provided below, sketch a continuous antiderivative $G(x)$ of $g(x)$ with domain $[-6, 6]$, satisfying $G(1) = 1$. Make sure to clearly label the input and output values at $x = -6, -1, 2, 4,$ and 6. Be sure to make it clear where $G(x)$ is **concave up**, **concave down**, or **linear**, and where it is **increasing**, **decreasing**, or not **differentiable**.



Solution: The input/output values at the specified points are labeled in the figure. The graph of $G(x)$ should be concave up on $(-6,-3.5)$, $(0,2)$, and $(4,6)$, concave down on $(-3.5,-1)$, and linear on $(-1,0)$ and $(2,4)$. The function $G(x)$ is increasing on $(-6,-1)$ and $(1,4)$ and decreasing on $(0,1)$ and $(4,6)$. The function $G(x)$ is not differentiable at $(0,1.5)$ and $(4,3.5)$.



8. [14 points] Let $g(x)$ be a differentiable function with domain $(-1, 10)$ where some values of $g(x)$ and $g'(x)$ are given in the table below. Assume that all local extrema and critical points of $g(x)$ occur at points given in the table.

x	0	1	2	3	4	5	6	7	8
$g(x)$	2.0	3.3	5.7	6.8	6.0	4.3	2.4	0.2	-4.9
$g'(x)$	2.8	2.5	2.0	0.0	-1.4	-1.9	-1.6	-3.0	-8.1

a. [3 points] Estimate $\int_0^8 g(x)\,dx$ using RIGHT(4). Write out each term in your sum.

Solution: With 4 rectangles the width of each is $\Delta x = \frac{8-0}{4} = 2$. Then

$$\begin{aligned} \text{RIGHT}(4) &= g(2)\Delta x + g(4)\Delta x + g(6)\Delta x + g(8)\Delta x \\ &= (5.7 + 6.0 + 2.4 - 4.9)\cdot 2 \\ &= 18.4 \end{aligned}$$

Answer: **18.4**

b. [4 points] Approximate the area of the region between $g(x)$ and the function $f(x) = x + 2$ for $0 \le x \le 4$, using MID(n) to estimate any integrals you use. Use the greatest number of subintervals possible, and write out each term in your sums.

Solution: The function $g(x)$ is concave down on $[0, 4]$, so $g(x)$ is greater than or equal to the linear function $f(x)$ on this interval. The integral to compute this area is

$$\int_0^4 g(x) - f(x)\,dx = \int_0^4 g(x)\,dx - \int_0^4 x + 2\,dx.$$

Since $f(x)$ is linear, we get the same answer whether we use MID to approximate $\int_0^4 g(x) - f(x)\,dx$ or just $\int_0^4 g(x)\,dx$ and compute $\int_0^4 f(x)\,dx$ exactly. In either case, we can use at most 2 subintervals and $\Delta x = 2$.
If we compute MID(2) for $\int_0^4 g(x) - f(x)\,dx$, we get

$$\begin{aligned} \text{MID}(2) &= (g(1) - f(1))\Delta x + (g(3) - f(3))\Delta x \\ &= ((3.3 - 3) + (6.8 - 5))2 \\ &= (.3 + 1.8)2 \\ &= 4.2. \end{aligned}$$

If we compute $\int_0^4 f(x)\,dx = 16$ and then compute MID(2) for $\int_0^4 g(x)\,dx$ we get

$$\begin{aligned} \text{MID}(2) &= g(1)\Delta x + g(3)\Delta x \\ &= (3.3 + 6.8)2 \\ &= 20.2. \end{aligned}$$

Then we get $20.2 - 16 = 4.2$ for the total area.

Answer: **4.2**

c. [3 points] Is your answer to **b.** an overestimate, an underestimate, or is there not enough information to tell? Briefly justify your answer.

Solution: Since we are only given a table and not told that the concavity does not change between points, we technically **do not have enough information** to answer this question.
Had it been the case that $g'(x)$ has no critical points aside from those in the table, it would follow that $g(x)$ is concave down, because $g'(x)$ would be decreasing on the given interval. Since $f(x)$ is linear, the concavity of $g(x) - f(x)$ would also be concave down. In that case, MID(2) would be an **overestimate.**
Credit was awarded for both of these answers.

d. [4 points] Write an integral giving the arc length of $y = g(x)$ between $x = 2$ and $x = 8$. Estimate this integral using TRAP(2). Write out each term in your sum.

Answer: Integral: $\int_2^8 \sqrt{1+g'(x)^2}\,dx$

Solution: The arc length is given by the integral

$$\text{Arc length} = \int_2^8 \sqrt{1+g'(x)^2}\,dx.$$

The width of our trapezoids is $\Delta x = \frac{8-2}{2} = 3$.
If we compute the areas of the trapezoids directly we get

$$\begin{aligned}\text{TRAP}(2) &= \left(\frac{\sqrt{1+g'(2)^2}+\sqrt{1+g'(5)^2}}{2}\right)\Delta x + \left(\frac{\sqrt{1+g'(5)^2}+\sqrt{1+g'(8)^2}}{2}\right)\Delta x \\ &\approx (2.1915795 + 5.1542930)3 \\ &\approx 22.0376175.\end{aligned}$$

If we compute LEFT(2) and RIGHT(2) first and then take an average we get

$$\begin{aligned}\text{LEFT}(2) &= \sqrt{1+g'(2)^2}\Delta x + \sqrt{1+g'(5)^2}\Delta x \\ &\approx (4.3831590)3 \\ &\approx 13.1494770.\end{aligned}$$

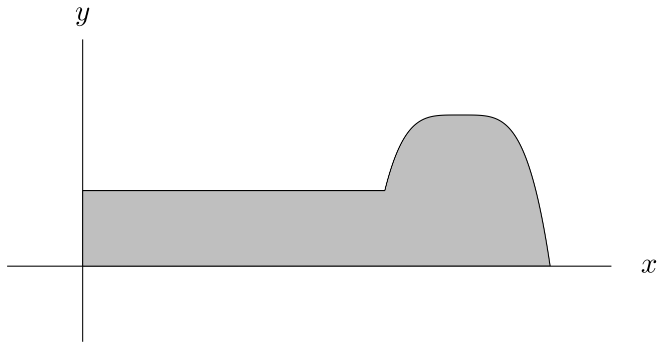
$$\begin{aligned}\text{RIGHT}(2) &= \sqrt{1+g'(5)^2}\Delta x + \sqrt{1+g'(8)^2}\Delta x \\ &\approx (10.3085860)3 \\ &\approx 30.9257580.\end{aligned}$$

Then

$$\text{TRAP}(2) = \frac{1}{2}(\text{LEFT}(2) + \text{RIGHT}(2)) \approx 22.0376175.$$

Answer: TRAP(2)= **22.0376175**

2. [13 points] Fred is designing a plastic bowl for his dog, Fido. Fred makes the bowl in the shape of a solid formed by rotating a region in the xy-plane around the y-axis. The region, shaded in the figure below, is bounded by the x-axis, the y-axis, the line $y = 1$ for $0 \le x \le 4$, and the curve $y = -(x-5)^4 + 2$ for $4 \le x \le 2^{1/4} + 5$. Assume the units of x and y are inches.



a. [7 points] Write an expression involving one or more integrals which gives the volume of plastic needed to make Fido's bowl. What are the units of your expression?

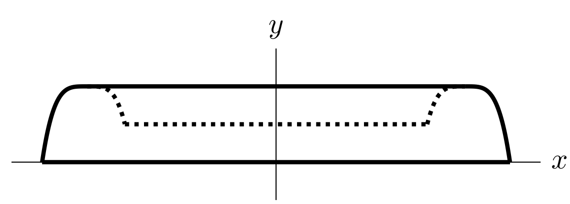
Solution: Using the cylindrical shell method, we have that the volume of plastic needed to make Fido's bowl is given by $\int_0^4 2\pi x dx + \int_4^{5+2^{\frac{1}{4}}} 2\pi x(2-(x-5)^4)dx$.

Using the washer method, we have that the volume of plastic needed to make Fido's bowl is given by $\pi \int_0^1 (5+(2-y)^{1/4})^2 dy + \pi \int_1^2 (5+(2-y)^{1/4})^2 - (5-(2-y)^{1/4})^2 dy$.

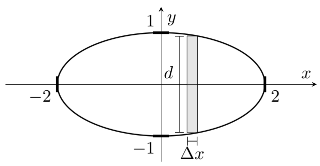
The units for either expression are in^3.

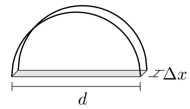
b. [6 points] Fred wants to wrap a ribbon around the bowl before he gives it to Fido as a gift. The figure below depicts the cross section of the bowl obtained by cutting it in half across its diameter. The thick solid curve is the ribbon running around this cross section, and the dotted curve is the outline of the cross section which is not in contact with the ribbon. Write an expression involving one or more integrals which gives the length of the thick solid curve in the figure (the length of ribbon Fred needs to wrap the bowl).

Solution: The length of ribbon Fred needs to wrap the bowl is given by
$10 + 2(2^{\frac{1}{4}} + 5) + 2\int_5^{5+2^{\frac{1}{4}}} \sqrt{1+16(x-5)^6}dx$.



8. [12 points] Astronomers have spotted a small near-Earth asteroid hurtling towards Earth. In order to assess its danger, they set about calculating its mass. Based on telescope images, the base of the asteroid is given by the region enclosed in the figure on the left, and its cross-sections perpendicular to the x-axis are semi-circles (as shown in the figure on the right). The base is the region bounded by $\frac{x^2}{4} + y^2 = 1$. A sample slice of the base of thickness Δx is shown in graph on the left, and all distances are given in meters.





a. [3 points] Write an expression for the diameter, d, in meters, of a cross-sectional slice of the asteroid x meters from the y-axis.

Answer: $d =$ $2\sqrt{1-\frac{x^2}{4}}$ **m**

b. [4 points] Write an expression for the volume, V, in m^3, of a cross-sectional slice of the asteroid x meters from the y-axis with thickness Δx meters.

Answer: $V =$ $\frac{\pi}{2}\left(1-\frac{x^2}{4}\right)\Delta x$ $\mathbf{m^3}$

c. [2 points] The density of the asteroid depends on x due to shearing (i.e. loss of material) in its direction of travel. The astronomers have computed the expression for the density of a cross-sectional slice of the asteroid to be $\delta(x) = \frac{4000}{7}(x+2)$ kg/m^3. What is the mass, $m(x)$, in kg, of a cross sectional slice of the asteroid with thickness Δx meters?

Answer: $m(x) =$ $\frac{\pi}{2}\left(1-\frac{x^2}{4}\right)\left(\frac{4000}{7}(x+2)\right)\Delta x$ **kg**

d. [3 points] Write an integral that gives the total mass of the asteroid in kg. Do not evaluate your integral.

Answer: Total Mass = $\int_{-2}^{2} \frac{\pi}{2}\left(1-\frac{x^2}{4}\right)\left(\frac{4000}{7}(x+2)\right)dx$ **kg**