Math 116 — Practice for Exam 2

Generated March 17, 2025

Name: **SOLUTIONS**

Instructor: ______________________ Section Number: __________

1. This exam has 9 questions. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
2. Do not separate the pages of the exam. If any pages do become separated, write your name on them and point them out to your instructor when you hand in the exam.
3. Please read the instructions for each individual exercise carefully. One of the skills being tested on this exam is your ability to interpret questions, so instructors will not answer questions about exam problems during the exam.
4. Show an appropriate amount of work (including appropriate explanation) for each exercise so that the graders can see not only the answer but also how you obtained it. Include units in your answers where appropriate.
5. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
6. If you use graphs or tables to obtain an answer, be certain to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
7. You must use the methods learned in this course to solve all problems.

Semester	Exam	Problem	Name	Points	Score
Winter 2010	3	2		18	
Winter 2013	1	6	swimming pool	11	
Fall 2018	2	2	bacteria parasite	9	
Winter 2018	2	7		10	
Winter 2024	2	6	song writing	10	
Fall 2023	3	8	vuvuzela	9	
Winter 2022	2	8		8	
Winter 2020	1	10		7	
Winter 2023	2	7	treasure	16	
Total				98	

Recommended time (based on points): 96 minutes

2. [18 points] For each of the following series, write whether the series "Converges" or "Diverges" on the space provided next to the series. Support your answer by stating the test(s) you used to prove convergence or divergence, and show complete work and justification.

a. [6 points] $\sum_{n=2}^{\infty} \frac{\sqrt{n^2+1}}{n^2-1}$ **Diverges**

Solution: We note that $\frac{1}{n^2-1} > \frac{1}{n^2}$ and $\sqrt{n^2+1} > n$ for $n \geq 2$, so $\frac{\sqrt{n^2+1}}{n^2-1} > \frac{1}{n}$. We know $\sum_{n=2}^{\infty} \frac{1}{n}$ diverges by the p-test, so by the comparison test, the series diverges.

b. [6 points] $\sum_{n=1}^{\infty} \frac{n!(n+1)!}{(2n)!}$ **Converges**

Solution: We use the ratio test to get

$$\lim_{n\to\infty} \frac{(2n)!(n+1)!(n+2)!}{(2(n+1))!(n+1)!n!}.$$

After some cancellation we get

$$\lim_{n\to\infty} \frac{(n+2)(n+1)}{(2n+1)(2n+2)} = 1/4.$$

Thus it converges.

c. [6 points] $\sum_{n=2}^{\infty} \frac{\sin(n)}{n^2-3}$ **Converges**

Solution: We will show that this is absolutely convergent, and hence convergent. Additionally, since $|\sin(n)| \leq 1$ we can bound the sum of the absolute values by

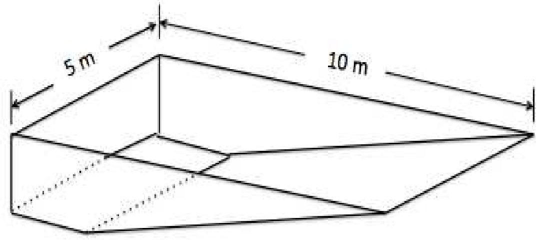
$$\sum_{n=2}^{\infty} \frac{1}{n^2-3}.$$

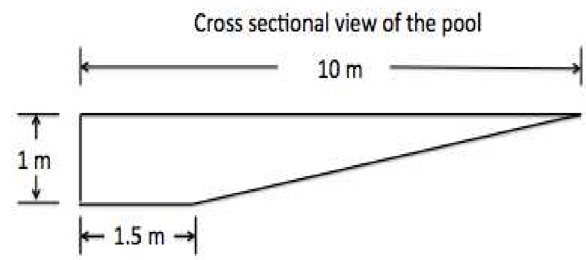
Then we apply the limit comparison test with $\sum_{n=2}^{\infty} \frac{1}{n^2}$. Since

$$\lim_{n\to\infty} \frac{n^2-3}{n^2} = 1$$

both series converge or both diverge. Since $\sum_{n=2}^{\infty} \frac{1}{n^2}$ converges by p-test, they both converge.

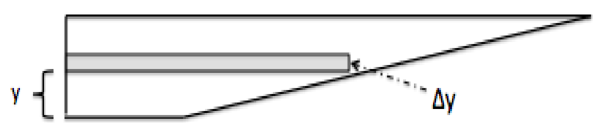
6. [11 points] A swimming pool 10 m long and 5 m wide has varying depth. Its maximum depth is 1 m as shown in the picture below





The swimming pool has water up to a level of maximum depth of 0.6 m. The density of water is 1000 kg per m^3. Use $g = 9.8$ m/s^2 for the acceleration due to gravity.

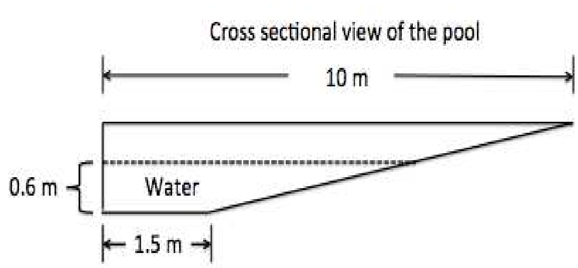
a. [9 points] Write an expression that approximates the work done in lifting a horizontal slice of water with thickness Δy meters, that is at a distance of y meters above the bottom, to the top of the swimming pool.



Solution: First we must find a formula for the length of the swimming pool at depth for a given height above the bottom. Let's call this function $l(y)$. We know that $l(0) = 1.5$ and $l(1) = 10$. Since $l(y)$ is a linear function, this tells us that $l(y) = 8.5y + 1.5$.
The volume of such a slice is $\Delta y(8.5y+1.5)\cdot 5$. Multiplying by 1000 kg/m^3 and 9.8 m/s^2 gives us the weight of the water in Newtowns. The amount the water needs to be lifted is $(1 - y)$. We therefore get:

$$W_{slice} \approx 1000 \cdot 9.8 \cdot (8.5y + 1.5) \cdot 5 \cdot (1 - y)\Delta y.$$

b. [2 points] Write a definite integral that computes the work required to pump all the water to the top of the pool.



Solution: Work $= \int_0^{0.6} 1000 \cdot 9.8 \cdot 5(8.5y + 1.5)(1 - y)dy$ Joules.

2. [9 points] Note: *"Closed form" here means that the expression should* NOT *include sigma notation or ellipses (...) and should* NOT *be recursive.*

Michel is studying how the mass of a certain collection of bacterial cells behaves in the presence of a parasite. He notices that from noon to midnight of each day, the parasite eats 60% of the mass of the bacterial cells. Then the parasite sleeps until noon the next day. While the parasite sleeps, the remaining 40% of the collection of bacterial cells doubles in mass.

At noon on the first day, the mass of the collection of bacterial cells is 100 grams.

a. [3 points] Let X_n be the mass, in grams, of bacterial cells present at noon on day n. Note that $X_1 = 100$. Calculate X_2 and X_3, and find a closed form expression for X_n.

n	X_n	Uneaten	After regrowth
1	100	$(.4)(100)$	$2(.4)(100) = 80 = (.8)(100)$
2	80	$(.4)(80)$	$2(.4)(80) = 64 = (.8)^2(100)$
3	64		

Answer: $X_2 =$ 80 g and $X_3 =$ 64 g

Answer: $X_n =$ $100(.8)^{n-1}$ g

b. [4 points] Let K_n be the total mass, in grams, of bacterial cells that the parasite has consumed in the first n days. For example, on day 1 the parasite consumes 60% of 100 grams, which is 60 grams, so $K_1 = 60$. Calculate K_2 and K_3, and find a closed form expression for K_n.

n	K_n
1	60
2	$60 + (.6)(80) = 108$
3	$108 + (.6)(64) = 146.4$

Amount eaten on day i is $(.6)X_i = (.6)(100)(.8)^{i-1} = 60(.8)^{i-1}$

so $K_n = \sum_{i=1}^{n} 60(.8)^{i-1} = 60(1 + (.8) + (.8)^2 + \cdots + (.8)^{n-1}) = 60\frac{1-(.8)^n}{1-.8}$

$= \frac{60}{.2}(1-(.8)^n) = 300(1-(.8)^n)$

Answer: $K_2 =$ 108 g and $K_3 =$ 146.4 g

Answer: $K_n =$ $300(1-(.8)^n)$ g

c. [2 points] If this continued forever, how many grams of bacterial cells would the parasite eventually eat?

$$\lim_{n\to\infty} 300(1-(.8)^n) = 300$$

Answer: Mass = 300 g

7. [10 points] Consider the two sequences a_n and b_n defined by

$$a_n = \frac{1}{2^n} \qquad\qquad b_0 = 5, \quad b_n = 3b_{n-1} \text{ for all } n > 1.$$

Compute the following limits. If the sequence diverges, write DIVERGES.
No justification necessary.

a. [2 points] $\lim_{n\to\infty} a_n$

Answer: $\lim_{n\to\infty} a_n =$ **0**

b. [2 points] $\lim_{n\to\infty} \sum_{k=0}^{n} a_k$

Answer: $\lim_{n\to\infty} \sum_{k=0}^{n} a_k =$ **2**

c. [2 points] $\lim_{n\to\infty} a_n b_n$

Answer: $\lim_{n\to\infty} a_n b_n =$ ∞ **or** DIVERGES

d. [2 points] $\lim_{n\to\infty} \frac{\ln(b_n)}{\ln(a_n)}$

Answer: $\lim_{n\to\infty} \frac{\ln(b_n)}{\ln(a_n)} =$ $\frac{\ln(3)}{\ln(1/2)} \approx -1.585$

e. [2 points] $\lim_{n\to\infty} \frac{1-e^{3a_n}}{a_n}$

Solution: Since this is a limit of the form 0/0, we solve this by applying L'Hôpital's Rule.

Answer: $\lim_{n\to\infty} \frac{1-e^{3a_n}}{a_n} =$ -3

6. [10 points] Liban is writing songs using a new style of music which he calls "new-age jazz." The longer that he spends writing a particular song, the better it turns out.

Let $Q(t)$ be the **cumulative distribution function** (cdf) for t, the number of days that it takes for Liban to write a particular song. The formula for $Q(t)$ is shown to the right, where $c > 0$ is a constant.

$$Q(t) = \begin{cases} 0 & t < 0, \\ \frac{c}{4}t^2 & 0 \le t \le 2, \\ 2c - c\,e^{2-t} & t > 2. \end{cases}$$

You do not need to show your work in this problem, but partial credit may be given for work shown.

a. [3 points] Write a piecewise-defined formula for $q(t)$, the **probability density function** (pdf) corresponding to $Q(t)$. Your answer may involve c, but it should not involve the letter Q.

Solution: We know that $Q(t)$ and $q(t)$ are related by the formula $Q'(t) = q(t)$. So, the formula for $q(t)$ is found by taking the derivative of each part of $Q(t)$.

$$q(t) = \begin{cases} 0 & t < 0, \\ \frac{c}{2}t & 0 \le t \le 2, \\ c\,e^{2-t} & t > 2. \end{cases}$$

b. [3 points] Write an expression involving one or more integrals that represents the **mean** number of days that it takes for Liban to write a particular song. Your answer may involve c, but it should not involve the letters Q or q. **Do not evaluate your integral(s).**

Solution: The formula for the mean is given by $\int_{-\infty}^{\infty} tq(t)\,dt$. Using our answer to part (a):

$$\int_{-\infty}^{\infty} tq(t)\,dt = \int_{-\infty}^{0} 0\,dt + \int_0^2 \frac{c}{2}t^2\,dt + \int_2^{\infty} c\,te^{2-t}\,dt = \int_0^2 \frac{c}{2}t^2\,dt + \int_2^{\infty} c\,te^{2-t}\,dt.$$

Answer: $\int_0^2 \frac{c}{2}t^2\,dt + \int_2^{\infty} c\,te^{2-t}\,dt$

c. [2 points] Use the fact that $Q(t)$ is a cumulative distribution function to find the value of c.

Solution: Since $Q(t)$ is a cumulative distribution function, we must have $\lim_{t\to\infty} Q(t) = 1$. Using the formula for $Q(t)$ for $t > 2$,

$$\lim_{t\to\infty} Q(t) = \lim_{t\to\infty}(2c - c\,e^{2-t}) = 2c.$$

Therefore $2c = 1$, so that $c = \frac{1}{2}$.

Answer: $c = \frac{1}{2}$

d. [2 points] Circle the **one** correct answer below that completes the following sentence:

"The quantity $Q(5)$ represents...

(i) *...the probability that it takes exactly 5 days for Liban to write a song."*

(ii) *...the probability that it takes more than 5 days for Liban to write a song."*

(iii) *...the probability that it takes 5 days or less for Liban to write a song."*

(iv) *...the approximate probability that it takes between 4.5 and 5.5 days for Liban to write a song."*

(v) NONE OF THESE

Solution: We know that $Q(5)$ and $q(5)$ are related by the formula $Q(5) = \int_{-\infty}^{5} q(t)\,dt$, and this integral represents the probability that it takes 5 days or less for Liban to write a song. To explain why the other choices are incorrect:

In general, the probability that it takes Liban between a and b days to write a song is the quantity

$$Q(b) - Q(a) = \int_a^b q(t)\,dt.$$

Thus the probability that it takes exactly 5 days for Liban to write a song must be

$$Q(5) - Q(5) = \int_5^5 q(t)\,dt = 0.$$

But $Q(5) \neq 0$ since we know $Q(t)$ is a nondecreasing function (as it is a cdf), and so we have $Q(5) \geq Q(2) = 0.5 > 0$, using our value of c from part (c). So (i) is incorrect.

The probability that it takes more than 5 days for Liban to write a song is $1 - Q(5) = \int_5^{\infty} q(t)\,dt$. We have $Q(2) = 0.5$. The formula for $Q(t)$ shows that it is strictly increasing, so $Q(5) > 0.5$, and thus $1 - Q(5) < 0.5$. This means $Q(5) \neq 1 - Q(5)$, so (ii) is incorrect.

The approximate probability that it takes between 4.5 and 5.5 days for Liban to write a song is a standard interpretation of the quantity $q(5)$, which describes a pdf, not a cdf. So (iv) is also incorrect.

8. [9 points] Gabriella is developing a new kind of vuvuzela. In order to come up with a new method, she first considers the old way she made her instruments.

a. [4 points] Gabriella initially made her vuvuzelas by considering a positive function $f(x)$, and forming a region $\mathcal{R}$ between $y = f(x)$ and the x-axis on the interval $[2, \infty)$. She rotated $\mathcal{R}$ about the x-axis to form the shape of the vuvuzela. Write an integral which gives the volume of the vuvuzela. Your answer will involve the function $f(x)$.

Answer: $\int_2^{\infty} \pi (f(x))^2 \, \mathrm{d}x$

b. [5 points] For her new batch of vuvuzelas, Gabriella considers an entirely different shape. The volume of the new design of vuvuzela is given by

$$\int_2^{\infty} \frac{x}{(x^2+5)^2} \, \mathrm{d}x.$$

Compute the value of this integral if it converges. If it does not converge, use a **direct computation** of the integral to show its divergence. Be sure to show your full computation, and be sure to use **proper notation**.

Solution: We begin by writing the improper integral as a corresponding limit. Using a substitution of $u = x^2 + 5$, we have

$$\begin{aligned}\int_2^{\infty} \frac{x}{(x^2+5)^2} \, \mathrm{d}x &= \lim_{b \to \infty} \int_2^{b} \frac{x}{(x^2+5)^2} \, \mathrm{d}x \\ &= \lim_{b \to \infty} \int_9^{b^2+5} \frac{(1/2)}{u^2} \, \mathrm{d}u \\ &= \lim_{b \to \infty} -\frac{1}{2(b^2+5)} + \frac{1}{2(9)} \\ &= -0 + \frac{1}{18}.\end{aligned}$$

Therefore the integral converges to $\frac{1}{18}$.

Circle one: **Diverges** **Converges to** $\frac{1}{18}$

8. [8 points] Determine whether the following improper integral converges or diverges. **Circle your final answer choice**. Fully justify your answer including using proper notation and showing mechanics of any tests you use.

$$\int_1^\infty \frac{t^2+\ln(t)}{t^3-\cos(t)+2}dt$$

Circle one: **Converges** **Diverges**

Solution: The numerator of the integrand is dominated by t^2, and the denominator is dominated by t^3, so this function has the same behavior as $\frac{t^2}{t^3} = \frac{1}{t}$, so we expect it to diverge. Therefore, we want to bound this function below by a function whose integral diverges. First, we note that $t^2 \le t^2 + \ln(t)$ on $[1, \infty)$. Then, for the denominator, since $\cos(x)$ oscillates from $[-1, 1]$, the denominator is largest(ans so the function is smallest) when $\cos(x) = -1$, so we get that $t^3 - \cos(t) + 2 \le t^3 + 1 + 2$, and so

$$\frac{t^2}{t^3+3} \le \frac{t^2}{t^3-\cos(t)+2} \le \frac{t^2+\ln(t)}{t^3-\cos(t)+2}$$

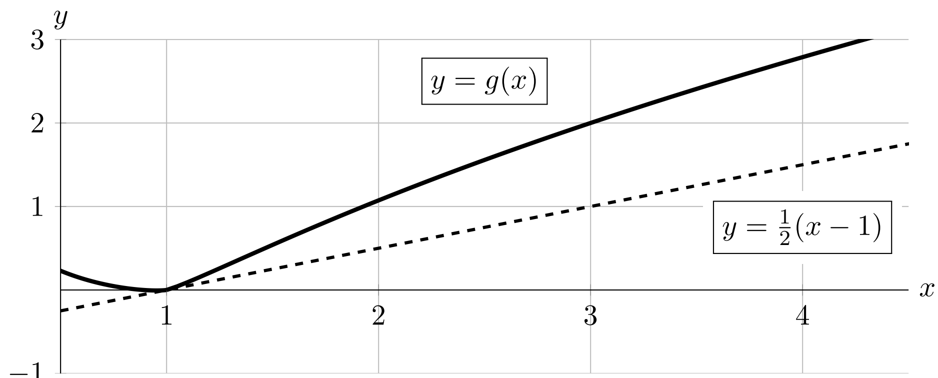
Next we know that $3 \le \frac{1}{2}t^3$ on $[2, \infty]$, and so $t^3 + 3 \le t^3 \frac{1}{2} t^3 = \left(\frac{3}{2}\right) t^3$, and so we get

$$\left(\frac{2}{3}\right)\frac{1}{t} = \left(\frac{2}{3}\right)\frac{t^2}{t^3} \le \frac{t^2}{t^3+3} \le \frac{t^2+\ln(t)}{t^3-\cos(t)+2}.$$

Then, $\frac{2}{3}\int_1^\infty \frac{1}{t}dt$ diverges by p-test, with $p = 1$, and so $\int_1^\infty \frac{t^2+\ln(t)}{t^3-\cos(t)+2}dt$ diverges by comparison test, comparing $\left(\frac{2}{3}\right)\frac{1}{t} \le \frac{t^2+\ln(t)}{t^3-\cos(t)+2}$

10. [7 points] Consider functions f and g that satisfy all of the following:

- $f(x)$ is defined, positive, and continuous for all $x > 1$.
- $\lim_{x \to 1^+} f(x) = \infty$ (so $f(x)$ has a vertical asymptote at $x = 1$).
- $g(x)$ is defined and differentiable for all real numbers x, and $g'(x)$ is continuous.
- $\frac{d}{dx}\left(\frac{g(x)}{\ln x}\right) = f(x)$ for all $x > 1$.
- The tangent line to $g(x)$ at $x = 1$ is given by the equation $y = \frac{1}{2}(x-1)$. Graphs of $g(x)$ (solid) and this tangent line (dashed) are shown below.



Determine whether the integral $\int_1^3 f(x)\,dx$ converges or diverges.

- If the integral converges, circle "Converges", find its exact value, and write the exact value on the answer blank provided.
- If the integral diverges, circle "Diverges" and carefully justify your answer.

Show every step of your work carefully, and make sure that you use correct notation.

Solution: Since $f(x)$ has a vertical asymptote at $x = 1$, we write

$$\begin{aligned}\int_1^3 f(x)\,dx &= \lim_{a\to 1^+}\int_a^3 f(x)\,dx\\ &= \lim_{a\to 1^+} \left.\frac{g(x)}{\ln x}\right|_a^3\\ &= \lim_{a\to 1^+}\left(\frac{g(3)}{\ln 3} - \frac{g(a)}{\ln a}\right)\\ &= \frac{2}{\ln 3} - \lim_{a\to 1^+}\frac{g(a)}{\ln a}\\ &= \frac{2}{\ln 3} - \lim_{a\to 1^+}\frac{g'(a)}{1/a} \quad \text{where we applied l'Hopital's Rule}\\ &= \frac{2}{\ln 3} - \frac{1/2}{1}\end{aligned}$$

So this improper integral converges.

Circle one:

$\int_1^3 f(x)\,dx$ **converges to** $\frac{2}{\ln 3} - \frac{1}{2}$ ______ or $\int_1^3 f(x)\,dx$ **diverges**

7. [16 points] A treasure hunter has spotted a large exotic rock at the bottom of a deep pit. The vertical distance from the top of the pit to the top of the rock is 15 meters. To retrieve the rock, the treasure hunter attaches a 15 meter rope to the top of the rock and lifts it out of the pit. The rope used has mass 2 kg per meter. Below, **do not simplify your final answers or evaluate any integrals**. As a reminder, the acceleration due to gravity is g, where $g = 9.8\,\mathrm{m/s^2}$.

a. [8 points] If the rock has mass 4 kg, write an expression involving integrals for the amount of work, in Joules, the treasure hunter does in lifting the rock and the attached rope 10 meters up from the bottom of the pit.

Hint: Once rope has been raised to the top of the pit, the treasure hunter no longer needs to lift it.

Solution: After lifting the rock and the attached rope a distance of h meters, the length of the attached rope yet to be lifted is $15 - h$ meters. Therefore, the combined mass of the rock and the attached rope being lifted at this moment is

$$4 + 2 \cdot (15 - h) \text{ kg.}$$

So, the work done to lift the rock and the attached rope a short distance Δh meters at this moment is,

$$(4 + 2 \cdot (15 - h)) \cdot 9.8 \cdot \Delta h \text{ Joules.}$$

Therefore, the amount of work done by the treasure hunter to lift the rock and the attached rope 10 meters up from the bottom of the pit is given by,

$$\int_0^{10} (4 + 2 \cdot (15 - h)) \cdot 9.8 \, dh \text{ Joules.}$$

Answer: $\int_0^{10} (4 + 2 \cdot (15 - h)) \cdot 9.8 \, dh$

b. [8 points] After the rock has been lifted 10 meters off the bottom of the pit, the rock starts to crumble, losing 0.1 kg of mass per second. The treasure hunter resumes lifting the rock at a constant speed of 0.5 meters per second. Write an expression involving integrals for the amount of work, in Joules, the treasure hunter does in lifting the crumbling rock (and the attached rope) the remaining 5 meters to the top of the pit.

*The hint from part **a.** still applies.*

Solution: Since the rock loses 0.1 kg of mass per second, and the treasure hunter lifts at a constant speed of 0.5 meters per second, we have that the rock loses mass at a rate of

$$\frac{0.1\frac{\text{kg}}{\text{s}}}{0.5\frac{\text{m}}{\text{s}}} = 0.2\frac{\text{kg}}{\text{m}}.$$

Now, after lifting the rock and the attached rope a further distance of x meters, the length of the attached rope yet to be lifted is $5 - x$ meters. Therefore, the combined mass of the rock and the attached rope being lifted at this moment is

$$(4 - 0.2x) + 2 \cdot (5 - x) \text{ kg}.$$

So, the work done to lift the rock and the attached rope a short distance Δx meters at this moment is,

$$((4 - 0.2x) + 2 \cdot (5 - x)) \cdot 9.8 \cdot \Delta x \text{ Joules}.$$

Therefore, the amount of work done by the treasure hunter to lift the crumbling rock (and the attached rope) the remaining 5 meters to the top of the pit is given by,

$$\int_0^5 ((4 - 0.2x) + 2 \cdot (5 - x)) \cdot 9.8 \, dx \text{ Joules}.$$

Alternatively, we can compute quantities after we have lifted the rock and the attached rope for a duration of t seconds. The mass of the crumbling rock is $4 - 0.1t$ kg. Now, the attached rope here starts of with a mass of $2 \cdot 5 = 10$ kg, and every second 0.5 meters of it is being retracted/lifted. So, the mass of the attached rope is given by $10 - 2(0.5 \cdot t) = 10 - t$ kg. The combined mass of the rock and the attached rope is then

$$(4 - 0.1t) + (10 - t) \text{ kg}.$$

Now, in the next short period of Δt seconds, the system is lifted $0.5\Delta t$ meters. So, the work done to lift the rock and the attached rope in this short period is

$$((4 - 0.1t) + (10 - t)) \cdot 9.8 \cdot 0.5\Delta t \text{ Joules}.$$

Now, at a speed of 0.5 meters per second, it takes 10 seconds to lift the system the remaining 5 meters to the top. Therefore, the amount of work done by the treasure hunter to lift the crumbling rock (and the attached rope) in this process is given by,

$$\int_0^{10} ((4 - 0.1t) + (10 - t)) \cdot 9.8 \cdot 0.5 \, dt \text{ Joules}.$$

Answer: $\int_0^5 ((4 - 0.2x) + 2 \cdot (5 - x)) \cdot 9.8 \, dx$ **or** $\int_0^{10} ((4 - 0.1t) + (10 - t)) \cdot 9.8 \cdot 0.5 \, dt$