Math 116 — Practice for Exam 2

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UMID: **SOLUTIONS** ______________________ Initials: ______

Instructor: ______________________ Section Number: ________

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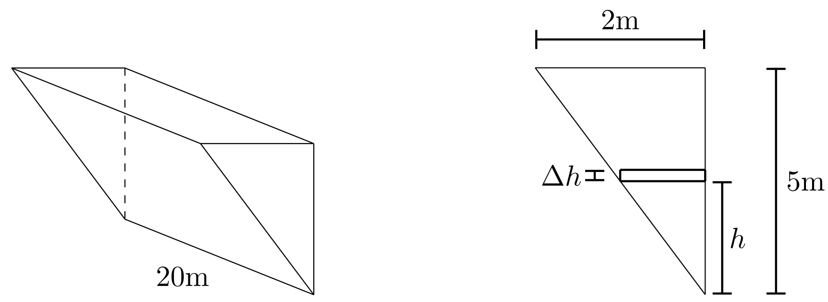
1. This exam has 9 questions. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
2. Please read the instructions for each individual exercise carefully. One of the skills being tested on this exam is your ability to interpret questions, so instructors will not answer questions about exam problems during the exam.
3. Show an appropriate amount of work (including appropriate explanation) for each exercise so that the graders can see not only the answer but also how you obtained it. Include units in your answers where appropriate.
4. You are allowed notes written on two sides of a $3'' \times 5''$ note card. You are NOT allowed other resources, including, but not limited to, notes, calculators or other electronic devices.
5. If you use graphs or tables to obtain an answer, be certain to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
6. Problems may ask for answers in exact form. Recall that $x = \sqrt{2}$ is a solution in exact form to the equation $x^2 = 2$, but $x = 1.41421356237$ is not.
7. You must use the methods learned in this course to solve all problems.

Semester	Exam	Problem	Name	Points	Score
Fall 2022	1	7	triangular tank	13	
Winter 2024	2	7		7	
Winter 2010	3	2		18	
Fall 2021	2	3	sheep	14	
Fall 2025	2	6		12	
Fall 2025	2	2	bus	9	
Winter 2021	2	10		12	
Winter 2018	3	4		11	
Winter 2022	2	8		8	
Total				104	

Recommended time (based on points): 102 minutes

7. [13 points] A drinking water facility needs to pump water out of an underground tank. The tank is 20 meters in length with right-triangular cross-sections perpendicular to the ground as shown in the figure. The top of the tank is a 2m by 20m rectangle. The **top** of the tank lies **5 meters below the surface of the earth.** Recall that $g = 9.8\text{m/s}^2$, where g is the gravitational constant.



Underground Tank

a. [5 points] Write an expression for the **volume** (in cubic meters) of a horizontal rectangular slice of water at height h above the bottom of the tank, with thickness Δh. Your answer should not involve an integral.

Solution: The length of the slice is 20m. Call the width w. To find w in terms of h, we use similar triangles (using the diagram on the right) to set up the proportion:

$$\frac{w}{h} = \frac{2}{5} \quad \Longrightarrow \quad w = \frac{2}{5}h.$$

Therefore the volume of such a slice is (we use that the volume of a rectangular prism is its length times its width times its height):

$$20 \cdot \frac{2}{5}h \cdot \Delta h = 8h\Delta h.$$

b. [2 points] The density of water is approximately 1000 kg/m^3. Write an expression for the **weight** (in Newtons) of the slice of water from part (a). Your answer should not involve an integral.

Solution: The density is constant, and therefore the mass of such a slice is the volume of that slice times 1000 kg/m^3, which is $8000h\Delta h$. To obtain the weight in Newtons, we multiply by $g = 9.8\text{m/s}^2$ to get $9.8 \cdot 8000h\Delta h$.

c. [3 points] Write an expression for the **work** (in Joules) needed to pump the slice of water (from parts (a) and (b)) to the surface of the earth. Your answer should not involve an integral.

Solution: (Note: the wording of this problem was slightly edited for clarity) The slice lies $5 - h$ meters below the top of the tank, and the top of the tank is 5 meters below the surface of the earth, so the total distance we move the slice up is $10 - h$ meters. Therefore the work (in Joules) done to move one slice up to the surface of the earth is approximately

$$(10 - h) \cdot (9.8 \cdot 8000h\Delta h).$$

d. [3 points] Assuming the tank is initially full of water, write an integral for the **total work** (in Joules) needed to pump all of the water to the surface of the earth.

Solution: Adding up the contributions of the work needed to move each slice found in part (c) and taking a limit as the thickness Δh of each slice goes to zero, we obtain the exact answer in the form of the integral

$$\int_0^5 (10 - h)(9.8 \cdot 8000h)\, dh.$$

7. [7 points] Determine whether the following series is convergent or divergent. **Fully justify** your answer including using **proper notation** and **showing mechanics** of any tests you use. Circle your final answer choice.

$$\sum_{n=2}^{\infty} \frac{4^n \cdot n^2}{n!}$$

Circle one: **Convergent** **Divergent**

Solution: We claim that the series is convergent. To show this, we use the ratio test. Let $a_n = \frac{4^n \cdot n^2}{n!}$. Then

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{\left|\frac{4^{n+1}\cdot(n+1)^2}{(n+1)!}\right|}{\left|\frac{4^n \cdot n^2}{n!}\right|} = \lim_{n\to\infty} \left|\frac{4^{n+1}\cdot(n+1)^2}{(n+1)!} \cdot \frac{n!}{4^n \cdot n^2}\right| = \lim_{n\to\infty} \frac{4(n+1)}{n^2} = 0 < 1.$$

Therefore, by the **ratio test,** the series $\sum_{n=2}^{\infty} \frac{4^n \cdot n^2}{n!}$ is **convergent.**

8. [6 points] Compute the following limit. **Fully justify** your answer including using **proper notation.**

$$\lim_{x\to 0} \frac{e^{2x} - (x+1)^2}{\cos(x) - 1}$$

Solution: We first note that this limit is of the indeterminate form $\frac{0}{0}$. We apply L'Hôpital's rule:

$$\lim_{x\to 0} \frac{e^{2x} - (x+1)^2}{\cos(x) - 1} \overset{L'H\ \frac{0}{0}}{=} \lim_{x\to 0} \frac{2e^{2x} - 2(x+1)}{-\sin(x)}.$$

But now this new limit is again of the indeterminate form $\frac{0}{0}$. We apply L'Hôpital's rule a second time:

$$\lim_{x\to 0} \frac{2e^{2x} - 2(x+1)}{-\sin(x)} \overset{L'H\ \frac{0}{0}}{=} \lim_{x\to 0} \frac{4e^{2x} - 2}{-\cos(x)} = \frac{4-2}{-1} = -2.$$

Therefore the original limit equals -2.

Answer: -2

2. [18 points] For each of the following series, write whether the series "Converges" or "Diverges" on the space provided next to the series. Support your answer by stating the test(s) you used to prove convergence or divergence, and show complete work and justification.

a. [6 points] $\sum_{n=2}^{\infty} \frac{\sqrt{n^2+1}}{n^2-1}$ ________ **Diverges**

Solution: We note that $\frac{1}{n^2-1} > \frac{1}{n^2}$ and $\sqrt{n^2+1} > n$ for $n \geq 2$, so $\frac{\sqrt{n^2+1}}{n^2-1} > \frac{1}{n}$. We know $\sum_{n=2}^{\infty} \frac{1}{n}$ diverges by the p-test, so by the comparison test, the series diverges.

b. [6 points] $\sum_{n=1}^{\infty} \frac{n!(n+1)!}{(2n)!}$ ________ **Converges**

Solution: We use the ratio test to get

$$\lim_{n\to\infty} \frac{(2n)!(n+1)!(n+2)!}{(2(n+1))!(n+1)!n!}.$$

After some cancellation we get

$$\lim_{n\to\infty} \frac{(n+2)(n+1)}{(2n+1)(2n+2)} = 1/4.$$

Thus it converges.

c. [6 points] $\sum_{n=2}^{\infty} \frac{\sin(n)}{n^2-3}$ ________ **Converges**

Solution: We will show that this is absolutely convergent, and hence convergent. Additionally, since $|\sin(n)| \leq 1$ we can bound the sum of the absolute values by

$$\sum_{n=2}^{\infty} \frac{1}{n^2-3}.$$

Then we apply the limit comparison test with $\sum_{n=2}^{\infty} \frac{1}{n^2}$. Since

$$\lim_{n\to\infty} \frac{n^2-3}{n^2} = 1$$

both series converge or both diverge. Since $\sum_{n=2}^{\infty} \frac{1}{n^2}$ converges by p-test, they both converge.

3. [14 points] Molly has recently become a sheep herder. She rotates her sheep through various fields so that the sheep have a varied diet and the fields have a chance to grow. Every Monday, the sheep visit the same field. Before the sheep graze for the first time in this field, its grass is 20 centimeters tall. Molly's sheep are picky and only eat the top 40% of the length of grass in this field every Monday. Over the course of the week, before the next Monday, the grass grows 3 centimeters. Let G_i represent the height in centimeters of the grass right before the sheep graze on it for the ith time. Note that $G_1 = 20$.

a. [5 points] Find expressions for each of G_2, G_3, and G_4. You do not need to evaluate your expressions.

Solution:

$$\begin{aligned} G_2 &= (0.6)G_1 + 3 \\ &= (0.6)(20) + 3 \\ G_3 &= (0.6)G_2 + 3 \\ &= (0.6)^2(20) + (0.6)(3) + 3 \\ G_4 &= (0.6)G_3 + 3 \\ &= (0.6)^3(20) + (0.6)^2(3) + (0.6)(3) + 3 \end{aligned}$$

b. [5 points] Find a general **closed-form** expression for G_n, defined for $n = 2, 3, 4 \ldots$

Solution:

$$\begin{aligned} G_n &= (0.6)^{n-1}(20) + \sum_{i=0}^{n-2} 3(0.6)^i \\ &= (0.6)^{n-1}(20) + \frac{3(1-(0.6)^{n-1})}{1-0.6} \end{aligned}$$

c. [4 points] In order for the field to meet sheep grazing standards, the height of the grass must be at least 5 cm when the sheep begin grazing. Molly thinks she will be able to stay on her field forever. Help her determine whether she can stay by either showing that the grass will eventually be less than 5 cm in height, or showing that the grass will be at least 5 cm each time before the sheep graze.

Solution:

$$\lim_{n\to\infty} G_n = \frac{3}{1-0.6} = 7.5.$$

Also note that G_n is a decreasing sequence. So, the grass is always taller than 5 cm. when the sheep begin grazing.

6. [12 points]

a. [6 points] For each of the following sequences or series described below, defined for $n \geq 1$, determine whether they must converge, must diverge, or whether there is not enough information. Circle your answers. No justification is required.

(i) $a_n = (-1)^n(2 + k^{-n})$, where k is a positive real number.

Circle one: **Converges** **Diverges** **Not Enough Information**

(ii) $b_n = \displaystyle\int_2^{n+3} f(x)\,dx$ where $f(x)$ is a positive function, and the series $\displaystyle\sum_{j=2}^{\infty} f(j)$ converges.

Circle one: **Converges** **Diverges** **Not Enough Information**

(iii) $c_n = P(e^n)$ where $P(x)$ is a cumulative distribution function.

Circle one: **Converges** **Diverges** **Not Enough Information**

b. [6 points] For each of the following sequences, defined for $n \geq 1$, decide whether the sequence is monotone increasing, monotone decreasing, or not monotone, and whether it is bounded or unbounded. Circle your answers. No justification is required.

(i) $r_n = \cos(2\pi n)\left(\dfrac{5}{4}\right)^n$

*Circle **all** which apply:*

Monotone Increasing **Monotone Decreasing** **Not Monotone**

Bounded **Unbounded**

(ii) $s_n = \dfrac{(-1)^n}{1 + \ln(n)}$

*Circle **all** which apply:*

Monotone Increasing **Monotone Decreasing** **Not Monotone**

Bounded **Unbounded**

(iii) $t_n = \displaystyle\int_1^{n^3} 2^{-x}\,dx$

*Circle **all** which apply:*

Monotone Increasing **Monotone Decreasing** **Not Monotone**

Bounded **Unbounded**

2. [9 points] Let t (in minutes) denote the time Audrey waits for the *Bursley-Baits* shuttle to arrive. Observations show that the **probability density function** (pdf) of her wait time (in minutes) is of the form

$$p(t) = \begin{cases} 0, & t < 0, \\ 2\lambda t e^{-\lambda t^2} & t \geq 0, \end{cases}$$

where λ is a positive constant.

Throughout this problem, show all your work, and write your answers in exact form.

a. [5 points] Suppose that Audrey's median wait time for the Bursley-Baits shuttle is 1 minute. Find the value of λ.

Solution: If we call the cumulative distribution function (cdf) $P(x)$, then

$$\begin{aligned} P(x) &= \int_0^x p(t)\, dt \\ &= \int_0^x 2\lambda t e^{-\lambda t^2}\, dt \\ &= \int_0^{\lambda x^2} e^{-u}\, du \\ &= 1 - e^{-\lambda x^2}. \end{aligned}$$

Therefore, solving $P(1) = 0.5$ we get $\lambda = \ln 2$.

Answer: $\ln 2$

Sometimes Audrey takes the *Northwood Express* shuttle. For the Northwood Express shuttle, Audrey's wait time (in minutes) follows a **cumulative distribution function** (cdf) of the form

$$Q(t) = \begin{cases} 0, & t < 0, \\ 1 - (\lambda t + 1)e^{-\lambda t} & t \geq 0, \end{cases}$$

where λ is the **same** as in part **a**.

b. [2 points] When Audrey takes the Northwood Express shuttle, what is the fraction of rides where Audrey waits for 1 minute or less for the shuttle to arrive? Your final answer should not involve λ.

Solution: We have

$$Q(1) = 1 - (\lambda + 1)e^{-\lambda} = \frac{1}{2} - \frac{1}{2}\ln 2.$$

Answer: $\frac{1}{2} - \frac{1}{2}\ln 2$

c. [2 points] Audrey wants to choose the shuttle that has a lower median wait time. Which one should she choose? Explain your answer.

Circle one: [THE BURSLEY-BAITS SHUTTLE] THE NORTHWOOD EXPRESS SHUTTLE

Explanation:

Solution: Since $Q(1) < 0.5$, we know that the median of the wait time for Northwood shuttle is greater than 1, while the median of the wait time for the Bursley-Baits shuttle is equal to 1. Therefore, Audrey should choose the Bursley-Baits shuttle.

10. [12 points] Show that the following statements are false by giving a concrete example to contradict each of the statement. You can write a formula or draw a clear, well-labeled graph in place of the blanks. Accompany your example with a brief but complete explanation.

a. [4 points] If $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=1}^{\infty} a_n$ converges.

Give your answer in the form:
"The statement is false when $a_n =$ ____________ because..."

Solution: For example, $\lim_{n\to\infty} \frac{1}{n} = 0$, but $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges by p-test, $p = 1$.

b. [4 points] For any continuous function $f(x)$ with $f(x) > 0$, the improper integral $\int_{-100}^{\infty} f(x)\,dx$ always diverges.

Give your answer in the form:

"The statement is false when $f(x) =$ ____________ because..."

Solution: An example is $f(x) = e^{-x}$, as $\int_{-100}^{\infty} e^{-x}\,dx = e^{100}$. We can also see that the integral converges by exponential decay test.

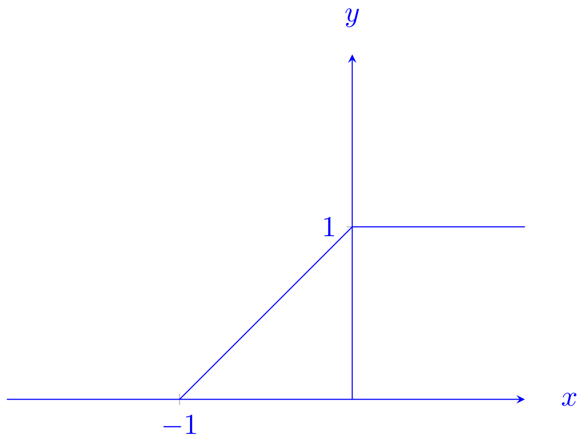
c. [4 points] If $P(x)$ is a cumulative distribution function, then $P(0) = 0$.

Give your answer in the form:

"The statement is false when $P(x) =$ ____________ because..."

(Note: Your $P(x)$ needs to be a cumulative distribution function, but you do not need to show/prove that it is.)

Solution: An example of $P(x)$ is given by the following graph.



In particular, $P(x)$ is indeed a cumulative distribution function, as $P(x)$ is increasing from 0 to 1 and it is continuous. However, $P(0) = 1 \neq 0$.

4. [11 points]

a. [6 points] Determine whether the following series converges absolutely, converges conditionally, or diverges, and give a complete argument justifying your answer.

$$\sum_{n=1}^{\infty}(-1)^n \sin\left(\tfrac{1}{n}\right)$$

Converges absolutely **Converges conditionally** **Diverges**

Justification:

Solution: This series converges by the alternating series test, which applies, since $\sin(\frac{1}{n})$ is a positive decreasing sequence that converges to zero.

It does not converge absolutely since for $n \geq 1$

$$\frac{1}{2n} \leq \sin\left(\tfrac{1}{n}\right).$$

We know the series $\sum_{n=1}^{\infty} \frac{1}{2n}$ diverges by p-test with $p = 1$. Then by the comparison test, so must $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right) = \sum_{n=1}^{\infty} |(-1)^n \sin\left(\frac{1}{n}\right)|$.

Alternatively, we can use the Limit Comparison Test. Since

$$\lim_{n\to\infty} \frac{\sin(1/n)}{1/n} = \lim_{x\to\infty} \frac{\sin(1/n)}{1/n} = \lim_{y\to 0} \frac{\sin(y)}{y} = 1 < 0,$$

we know that $\sum_{n=1}^{\infty} \frac{1}{n}$ and $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$ must either both converge or both diverge. Since $\sum_{n=1}^{\infty} \frac{1}{n}$ is the harmonic series, which we know diverges, $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$ must diverge as well.

b. [5 points] Compute the value of the following improper integral. **Show all your work using correct notation.** Evaluation of integrals must be done **without a calculator.**

$$\int_0^\infty \frac{e^x}{(1+e^x)^2}\,dx$$

Solution:
First we change to limit notation, then use u-substitution with $u = 1 + e^x$.

$$\begin{aligned}\int_0^\infty \frac{e^x}{(1+e^x)^2}\,dx &= \lim_{b\to\infty}\int_0^b \frac{e^x}{(1+e^x)^2}\,dx\\ &= \lim_{b\to\infty}\int_2^b \frac{1}{u^2}\,du\\ &= \lim_{b\to\infty} -\frac{1}{u}\Big|_2^b\\ &= \lim_{b\to\infty} \frac{-1}{b} - \frac{-1}{2} = \frac{1}{2}\end{aligned}$$

Alternatively, first compute the antiderivative using u-substitution.

$$\int \frac{e^x}{(1+e^x)^2}\,dx = \int \frac{1}{u^2}\,du = -\frac{1}{u} = -\frac{1}{1+e^x}.$$

Thus,

$$\int_0^\infty \frac{e^x}{(1+e^x)^2}\,dx = \lim_{b\to\infty}\int_0^b \frac{e^x}{(1+e^x)^2}\,dx = \lim_{b\to\infty} -\frac{1}{1+e^b} + \frac{1}{2} = \frac{1}{2}.$$

8. [8 points] Determine whether the following improper integral converges or diverges. **Circle your final answer choice**. Fully justify your answer including using proper notation and showing mechanics of any tests you use.

$$\int_1^\infty \frac{t^2 + \ln(t)}{t^3 - \cos(t) + 2} dt$$

Circle one: **Converges** **Diverges**

Solution: The numerator of the integrand is dominated by t^2, and the denominator is dominated by t^3, so this function has the same behavior as $\frac{t^2}{t^3} = \frac{1}{t}$, so we expect it to diverge. Therefore, we want to bound this function below by a function whose integral diverges. First, we note that $t^2 \leq t^2 + \ln(t)$ on $[1, \infty)$. Then, for the denominator, since $\cos(x)$ oscillates from $[-1, 1]$, the denominator is largest(ans so the function is smallest) when $\cos(x) = -1$, so we get that $t^3 - \cos(t) + 2 \leq t^3 + 1 + 2$, and so

$$\frac{t^2}{t^3 + 3} \leq \frac{t^2}{t^3 - \cos(t) + 2} \leq \frac{t^2 + \ln(t)}{t^3 - \cos(t) + 2}$$

Next we know that $3 \leq \frac{1}{2}t^3$ on $[2, \infty]$, and so $t^3 + 3 \leq t^3 \frac{1}{2} t^3 = \left(\frac{3}{2}\right) t^3$, and so we get

$$\left(\frac{2}{3}\right) \frac{1}{t} = \left(\frac{2}{3}\right) \frac{t^2}{t^3} \leq \frac{t^2}{t^3 + 3} \leq \frac{t^2 + \ln(t)}{t^3 - \cos(t) + 2}.$$

Then, $\frac{2}{3} \int_1^\infty \frac{1}{t} dt$ diverges by p-test, with $p = 1$, and so $\int_1^\infty \frac{t^2 + \ln(t)}{t^3 - \cos(t) + 2} dt$ diverges by comparison test, comparing $\left(\frac{2}{3}\right) \frac{1}{t} \leq \frac{t^2 + \ln(t)}{t^3 - \cos(t) + 2}$