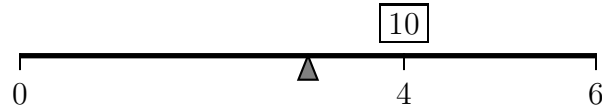


Torque, Center of Mass, and Moment

Here's an attempt to explain what center of mass is. We need a little bit of physics.



Suppose we have a 6-foot board with a 10-pound bag of flour on it. If we put a pivot under the board, the flour will create a **torque** on the board. Torque is like force, but it tends to rotate things instead of moving them in a straight line. The torque generated by the flour is

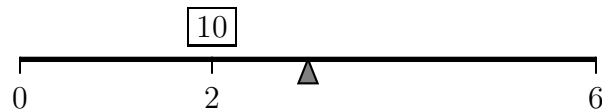
$$T = r \times F$$

where F is the force that gravity exerts on the flour (i.e., 10 lb), and r is the distance from the pivot to the bag. So if the pivot is in the middle of the board ($x = 3$ ft) and the flour is a foot to the right ($x = 4$ ft) then the torque is

$$(4 \text{ ft} - 3 \text{ ft}) \times 10 \text{ lb} = 10 \text{ ft}\cdot\text{lb}.$$

The greater a torque is, the more the board will tend to rotate. This explains how a wrench works: you put a long wrench on a stuck bolt and push, and you generate a torque which is greater (because of the length of the wrench) than if you just tried to turn the bolt with your hand.

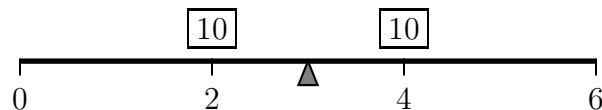
The board above will tend to rotate clockwise. If, instead of the flour on the right, we put a 10 pound block of ice to the left of the pivot at $x = 2$, like this:



then the board will tend to rotate counterclockwise. To keep the two cases straight, we'll say that the flour was generating a torque of $+10 \text{ ft}\cdot\text{lb}$ and the ice is generating a torque of $-10 \text{ ft}\cdot\text{lb}$. Positive torque means clockwise, negative means counterclockwise. We can keep our simple formula " $T = r \times F$ " by making r negative when the weight is to the left of the pivot and positive when it's to the right. That's what we get when we set

$$r = (x\text{-position of weight}) - (x\text{-position of pivot}).$$

Having signed torques is convenient, because if we have both the flour and ice on the board,

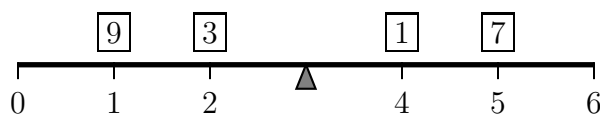


then we expect the board to balance. And indeed, we have

$$\text{Total torque} = \text{flour torque} + \text{ice torque} = (+10 \text{ ft}\cdot\text{lb}) + (-10 \text{ ft}\cdot\text{lb}) = 0$$

and a torque of 0 means no rotation.

Now it's easy to see how to generalize this to lots of weights—we just add up all the individual torques and we get the total torque. So for instance if we have something like this:



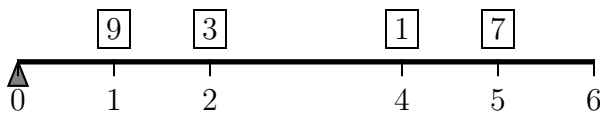
then the total torque on the board is

$$\begin{aligned} T &= (1 - 3) \times 9 + (2 - 3) \times 3 + (4 - 3) \times 1 + (5 - 3) \times 7 \\ &= (-18) + (-3) + (1) + (14) = -6 \text{ ft}\cdot\text{lb}. \end{aligned}$$

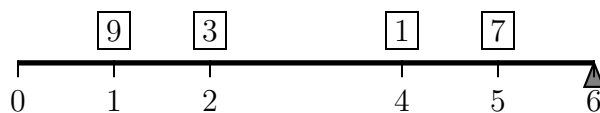
In general, if the pivot is at x -position a and there are weights w_1, w_2, \dots, w_n at positions x_1, x_2, \dots, x_n then the total torque is

$$T = (x_1 - a)w_1 + (x_2 - a)w_2 + \dots + (x_n - a)w_n.$$

Clearly if there are weights on the board and we put the pivot all the way at the left, the board will rotate clockwise (i.e., there is positive torque).



And if we place the pivot all the way on the right, the board will rotate counterclockwise (negative torque).



So by the Intermediate Value Theorem, there should be someplace in the middle where we can put the pivot and have the whole thing balance just right (zero torque). (Some people call this a “Three Bears” type of proof.)

We call that position \bar{x} , the **center of mass**. If we put \bar{x} in for a in the equation above, we should get 0 torque. That is,

$$\begin{aligned} 0 = T &= (x_1 - \bar{x})w_1 + (x_2 - \bar{x})w_2 + \dots + (x_n - \bar{x})w_n \\ &= (x_1w_1 + x_2w_2 + \dots + x_nw_n) - \bar{x}(w_1 + w_2 + \dots + w_n). \end{aligned}$$

If we solve that equation for \bar{x} we get

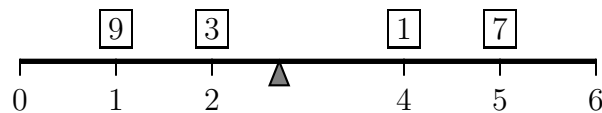
$$(1) \quad \boxed{\bar{x} = \frac{x_1 w_1 + x_2 w_2 + \cdots + x_n w_n}{w_1 + w_2 + \cdots + w_n}}$$

The denominator is just the total amount of weight on the board. The numerator we call the **moment** of the weights. Why we call it that is lost to the sands of time, but it comes up in many different mathematical and physical contexts.

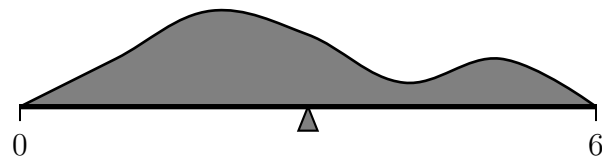
In the case of our example above, we have

$$\bar{x} = \frac{(1)(9) + (2)(3) + (4)(1) + (5)(7)}{9 + 3 + 1 + 7} = \frac{54}{20} = 2.7.$$

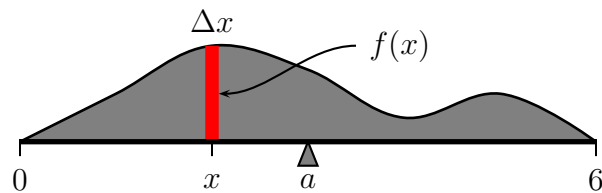
So if we place the pivot at position 2.7, the board should balance. (Check it!)



Now Suppose that instead of a few discrete weights, we have a pile of sand on the board, and its height varies according to some function $f(x)$:



You can imagine a vertical slice of sand as a weight:



The slice has weight

$$\Delta F = k f(x) \Delta x$$

where k is a constant representing the weight density of the sand. With the pivot at position a , the slice generates a torque equal to

$$\Delta T = r \times \Delta F = (x - a) k f(x) \Delta x,$$

so the total torque is

$$T = k \int_0^6 (x - a) f(x) dx.$$

Again there should be a center of mass \bar{x} , and we find it by setting the torque equal to 0:

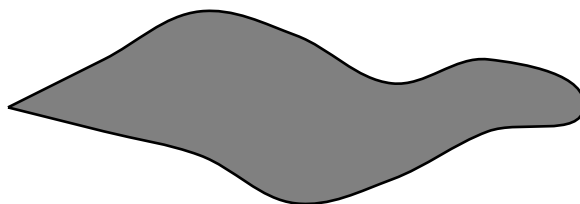
$$0 = T = k \int_0^6 (x - \bar{x})f(x) dx = k \int_0^6 xf(x) dx - k\bar{x} \int_0^6 f(x) dx.$$

Just like before, we can solve for \bar{x} and get

$$(2) \quad \boxed{\bar{x} = \frac{\int_0^6 xf(x) dx}{\int_0^6 f(x) dx}}.$$

The bottom is the total amount of sand, and the top is the moment of the sand. Note the similarity between equations (1) and (2). When you change from the discrete case (isolated weights) to the continuous case, sums become integrals, but the principles remain the same.

The examples with the board were all essentially one-dimensional, since the position of the pivot is just one number. But you might have some two-dimensional region, and want to find the center of mass of that.



You can find \bar{x} just like before, by finding vertical slices, and summing as x goes from left to right. So

$$\bar{x} = \frac{1}{A} \int xh(x) dx$$

where A is the total area of the region and $h(x)$ is the height of the slice at horizontal position x . Then you can find \bar{y} by taking horizontal slices and summing as y goes from bottom to top, so that

$$\bar{y} = \frac{1}{A} \int yw(y) dy$$

where $w(y)$ is the width at vertical position y . What's remarkable (and not obvious, at least not to me) is that the point (\bar{x}, \bar{y}) is really the center of mass of the region - if you put a pivot wedge (a triangular prism) under the region, it will balance if and only if the wedge goes through the center of mass, *regardless of the angle of the wedge*. Pretty cool!

For example, on the campus of Penn State University is a marker which says that it's at the exact center of mass of the state of Pennsylvania. So if you put a wedge under the state, it would balance if and only if the wedge went under Penn State.

We'll have more examples of using moments when we talk about probability densities after the exam.